A Continuous Time Framework for Sequential Goal-Based Wealth Management

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3 SGBWM for Linear Preferences

4 SGBWM For General Preferences

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Beyond Portfolio Volatility

- Most academic research associates risk with the volatility of an investor’s portfolio

- However, most investors associate risk with the probability of not attaining their goals

**Important distinction between goal risk and volatility:**

- Decreasing standard deviation in an underfunded investor’s portfolio increases likelihood of not attaining goals
Goal-Based Investing

Quoting Robert Merton:

“Goal-based investing will be very important in the next decade. For example, if you have a goal of funding retirement or a benefit plan, you set the goal and manage it through a process called LDI (liability-driven investing). If you follow a liability-driven goal, then regardless of whether your Sharpe ratio exceeds those of your competitors, you can outperform competitors who lose their focus on the goal. . . . We will be driven to the idea of greater service by knowing the client better, understanding what the client really needs, getting the client to identify what the actual goal is, and then designing dynamic strategies that achieve that goal.”
Different Tiers of Investment Goals

Figure: Different tiers of investment goals based on degree of urgency
Investment Goals

<table>
<thead>
<tr>
<th>Response</th>
<th>Percentage</th>
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<tbody>
<tr>
<td>Retirement</td>
<td>46%</td>
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<tr>
<td>Planning a vacation</td>
<td>43%</td>
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<tr>
<td>Emergencies</td>
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<tr>
<td>Purchase a new home</td>
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<tr>
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<td>29%</td>
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<tr>
<td>Major renovation</td>
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<tr>
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<td>Inheritance for my family</td>
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<td>Luxury purchases</td>
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<td>Philanthropy</td>
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<tr>
<td>Other investment goals</td>
<td>24%</td>
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</table>

**Figure:** Source: Franklin Templeton
Goal-based wealth management focuses around terms familiar to households.

The two largest robo-advising firms, Schwab and Betterment, provide services using a goal-based strategy:

- For each goal type, Betterment provides a maximum and minimum recommended stock allocation.

Social psychology shows that agents are intrinsically driven by goals in their daily life (Locke and Latham 2002).
Handling Multiple Goals

- How to prioritize between **multiple goals**?

- Should an investor forgo an immediate goal in order to increase the chances of attaining future goals?
  - Relative importance of goals
  - Partial fulfillment of goals (e.g., buy a cheaper house)
  - All-or-nothing goals (e.g., upgrades and renovations)

- Efficiency gains from optimizing all goals in a single portfolio
  - Enables offsets between underfunded and overfunded goals
Related Literature

- Das et al. (2018, 2020, 2021): Static and dynamic models of wealth management; finitely many portfolios on the efficient frontier


- Cvitanic et al. (2020): weighted average of probabilities of achieving target levels and avoiding specified loss levels; single deadline

- Capponi et al. (2020): personalized robo-advising with dynamic mean-variance risk preferences
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A $d$-dimensional stochastic factor process $Y = (Y_1, \ldots, Y_d)^\top$, whose dynamics is given by

$$dY_t = \mu_Y(Y_t)dt + \Sigma_Y(Y_t)dW_t,$$

where $\mu_Y : \mathbb{R}^d \to \mathbb{R}^d$ and $\Sigma_Y : \mathbb{R}^d \to \mathbb{R}^{d \times N'}$ are assumed to be Lipschitz continuous.

- Factor process captures macroeconomic factors (e.g. inflation, employment, and gross domestic product).
- Factor process drives the asset dynamics and influences client’s specifications.
Market Dynamics

- Investment in $N$ risky assets with dynamics

$$dS_{i,t} = S_{i,t}(\mu_i(Y_t)dt + \sigma_i(Y_t)dW_t), \quad i = 1, \ldots, N,$$

where $\mu(y)$ and $\Sigma(y) = (\sigma_1(y), \ldots, \sigma_N(y))^\top$ are bounded Lipschitz

- Lipschitz property of $\mu_Y, \Sigma_Y$ and bounded Lipschitz property of $\mu, \Sigma$

imply existence of a unique strong solution to the coupled system of asset and factor equation

- Assume an arbitrage-free market: $N' \geq N$ and $\Sigma\Sigma^\top$ assumed to be invertible
Client’s Input

Client provides the following input:

- Initial capital $x_0$
- Factor-dependent income stream function $I(\cdot) \geq 0$
  - E.g., fraction of client’s paycheck added to her portfolio
- Goal characteristics:
  - $G_k(\cdot)$ is the amount of $k$-th goal, and depends on the factor process
  - $T_k$ is the deadline of $k$-th goal, $T_1 < T_2 \cdots < T_M$
  - $\alpha_k$ is the importance weight of the $k$-th goal, $\sum_{k=1}^{M} \alpha_k = 1$
- Maximal portfolio volatility $c(\cdot)$ is a function of the factor process
Dynamics of client’s wealth $X(t)$ between consecutive goal deadlines:

$$dX_t = I(Y_t)dt + rX_t \left(1 - \sum_{i=1}^{N} \pi_{i,t}\right) dt + \sum_{i=1}^{N} X_t \pi_{i,t} \left(\mu_i(Y_t)dt + \sigma_i(Y_t)dW_t\right)$$

where $\pi(t) = (\pi_0(t), \ldots, \pi_N(t))$ is the fraction of wealth invested in each asset.

At goal deadline $T_k$, the amount $G_k \theta_k$ is withdrawn by the client:

$$X_{T_k} = X_{T_k^-} - G_k(Y_{T_k})\theta_k.$$

$\theta_k \in [0, 1]$ is the **funding ratio** for the $k$-th goal:

- $\theta_k = 1$: the $k$-th goal is fully funded
- $\theta_k < 1$: the $k$-th goal is partially funded with funding ratio $\theta_k$
Define $\Delta := \{ \pi \in \mathbb{R}^N_+ : \pi \mathbb{1}_N \leq 1 \}$ to be the $N$-dimensional simplex. $\pi_t$ takes values in the convex set $\Delta_c(Y_t)$, where

$$\Delta_c(y) := \Delta \cap \{ \pi \in \mathbb{R}^N_+ : ||\pi \Sigma(y)|| \leq c(y) \}$$

**Definition**

The set of **admissible controls** for the problem starting at time $t$ with wealth $x$ and factor value $y$, denoted by $\mathcal{A}_t(x, y)$, consists of the pairs $(\pi, \theta)$ where $\pi$ is a progressively measurable, $\Delta_c(Y_t)$-valued process, and $\theta$ is a random vector whose $k$-th component is $\mathcal{F}_{T_k}$-measurable, $[0, 1]$-valued, and satisfies $G_k(Y_{t_k}^{t,y})\theta_k \leq X_{T_k}^{t,x,y,\pi,\theta}$ for all $k$ such that $T_k > t$. 
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Linear Preferences

- Objective criterion: **expected weighted goal fundedness**, i.e., choose \( \pi = (\pi_1, \ldots, \pi_N) \) and \( \theta = (\theta_1, \ldots, \theta_M) \) to maximize

\[
\mathbb{E} \left[ \sum_{k=1}^{M} \alpha_k \theta_k \mid X_0 = x_0, Y_0 = y_0 \right],
\]

- Equivalent to minimizing expected shortfall \( \sum_{k=1}^{M} \alpha_k \mathbb{E}[(1 - \theta_k)^+] \)

- Value function

\[ V(t, x, y) := \sup_{(\pi, \theta) \in \mathcal{A}_t(x, y)} \mathbb{E}_{t, x, y} \left[ \sum_{k=1}^{M} \alpha_k \theta_k 1\{T_k > t\} \right], \]
Optimal Fundedness at Goal Deadlines

- Last goal deadline: \( V(T_M, x, y) = 0 \)
- At intermediate goal deadlines, solve a **static optimization** problem:
  \[
  V(T_k^-, x, y) = \lim_{t \uparrow T_k} V(t, x, y) = \sup_{\theta_k} \{ \alpha_k \theta_k + V(T_k, x - G_k(y)\theta_k, y) \}
  \]
  such that \( \theta_k \in [0, 1] \), and \( G_k(y)\theta_k \leq x \).
- At the last goal deadline
  - \( \theta^*_M(x, y) = 1 \wedge \frac{x}{G_M(y)} \)
  - \( V(T_M^-, x, y) = \alpha_M \left( 1 \wedge \frac{x}{G_M(y)} \right) \)
  \( \texttt{non differentiable} \)
Value Function Between Goal Deadlines

For \((t, x, y) \in [T_{k-1}, T_k) \times \mathbb{R}_+ \times \mathbb{R}^d\), we expect the value function to satisfy the following Hamilton-Jacobi-Bellman (HJB)

\[
V_t + \sup_{\pi \in \Delta_c(y)} \left\{ V_x(I(y) + rx + x\pi(\mu(y) - r\mathbb{1}_N)) + x\pi\Sigma\Sigma^T_Y(y)V_{xy} + \frac{1}{2} V_{xx}x^2\|\pi\Sigma(y)\|^2 + \mu^T_Y(y)V_y + \frac{1}{2} \text{Tr}(\Sigma_Y\Sigma^T_Y(y)V_{yy}) \right\} = 0
\]

with \textit{continuous, but not differential}, terminal condition \(V(T_{k-1}, x, y)\)
Characterization of Optimal Funding Ratio

Theorem

Assume $V(t, x, y)$ is concave in $x$. Any optimizer can be characterized by the generalized version of the Karush-Kuhn-Tucker (KKT) conditions. The largest optimizer is given in feedback form by

$$
\theta^*_k(x, y) = \inf \left\{ \theta \in \left[0, \frac{x}{G_k(y)}\right] : \partial^+_x V(T_k, x - G_k(y)\theta, y) > \frac{\alpha_k}{G_k(y)} \right\}
$$

\wedge 1 \wedge \frac{x}{G_k(y)}.

It is optimal to increase $\theta_k$ from zero until either

1. Goal $k$ is fully funded: $\theta_k(y) = 1$
2. Budget constraint is tight: $\theta_k(y)G_k(y) = x$
3. Marginal benefit $\frac{\alpha_k}{G_k(y)}$ of funding the current goal is smaller than that of saving for future goal liabilities $\partial^+_x V(T_k, x - G_k(y)\theta, y)$
Structure of Optimal Funding Ratio

Corollary

Fix $y \in \mathbb{R}^d$. The largest optimizer $\theta^*_k(\cdot, y)$ is piecewise linear and given by

$$
\theta^*_k(x, y) = \begin{cases} 0, & \text{if } 0 \leq x \leq b_k(y), \\
1 \wedge \frac{x-b_k}{G_k(y)}, & \text{if } x > b_k(y),
\end{cases}
$$

where $b_k(y) := \sup\{x \geq 0 : \partial^+_x V(T_k, x, y) > \alpha_k / G_k(y)\} \lor 0$, with the convention that $\sup\emptyset = -\infty$.

- $b_k$ is the “consumption threshold” for the $k$-th goal.
Funding vs. Saving Tradeoff

Proposition

(i) \( b_k(y) \leq s(T_k, y) \), where \( s(t, y) \) is the safe level

(ii) If \( \frac{\alpha_k}{G_k(y)} \leq \max_{j > k} \mathbb{E}[\alpha_j/G_j(Y_{T_k, y}^{T_j})] \) and \( \mathbb{E}\left[\int_{T_k}^{T_M} I(Y_{s,T_k,y}^{T_k})ds\right] \) is sufficiently small, then \( b_k(y) > 0 \), where \( Y_{T_k,y}^{T_k} \) denotes the factor process \( Y \) starting at \( y \) at time \( T_k \).

- No benefit in saving if current wealth equals or exceeds the safe level
- Save if some upcoming goals have a high priority or low amounts, and future income streams are small
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General Preferences

• Choose $\pi = (\pi_1, \ldots, \pi_N)$ and $\theta = (\theta_1, \ldots, \theta_M)$ to maximize

$$\max_{\pi, \theta} \mathbb{E} \left[ \sum_{k=1}^{M} \alpha_k f_k(\theta_k) \right],$$

where $f_k(\theta)$ quantifies the client’s satisfaction rate if the fundedness of the $k$-th goal is $\theta$

• **Examples:**
  - Linear Preference: $f_k(\theta) = \theta$
  - All-or-Nothing Preference: $f_k(\theta) = 1_{\theta=1}$
Value Function

- Value function defined by

\[ V(t, x, y) := \sup_{(\pi, \theta) \in \mathcal{A}_t(x, y)} \mathbb{E}_{t, x, y} \left[ \sum_{k=1}^{M} \alpha_k f_k(\theta_k) 1_{\{T_k > t\}} \right] \]

on the extended domain \( \overline{S} = [0, T_M] \times \mathbb{R} \times \mathbb{R}^d \)

- For a locally bounded function \( g : \mathcal{O} \rightarrow \mathbb{R} \), denote by

\[ g^*(z) := \limsup_{z' \to z, z' \in \mathcal{O}} g(z) \quad \text{and} \quad g_*^*(z) := \liminf_{z' \to z, z \in \mathcal{O}} g(z) \]

the upper and lower semicontinuous envelopes of \( g \) in \( \mathcal{O} \).
Value Functions and Optimal Funding Ratios

**Theorem**

Suppose \( f_k : [0, 1] \rightarrow [0, 1] \) is continuous. In the extended domain \( \bar{S} \):

(a) \( g_k \) is a continuous function of \( x \) and \( y \):

\[
g_k(x, y) := \sup_{\theta_k} \{ \alpha_k f_k(\theta_k) + V(T_k, x - G_k(y)\theta_k, y) : \\
\theta_k \in [0, 1], G_k(y)\theta_k \leq x^+ \}
\]

(b) The largest maximizer \( \theta_k^*(x, y) \) exists and is upper semicontinuous.

(c) We have \( V^*(T_k, x, y) = V_*(T_k, x, y) = g_k(x, y) \)

\( S_k := [T_{k-1}, T_k) \times \mathbb{R} \times \mathbb{R}^d \). In particular, \( V(T_k-, x, y) = g_k(x, y) \).

(d) \( V \) is continuous on \( S_k \), and is the unique viscosity solution to:

\[
- \nu_t + F(x, y, \nu_x, \nu_y, \nu_{xy}, \nu_{xx}, \nu_{yy}) = 0, \quad (t, x, y) \in S_k,
\]

\[
\nu(T_k, x, y) = g_k(x, y), \quad (x, y) \in \mathbb{R} \times \mathbb{R}^d,
\]

where

\[
F(x, y, p, q, g, A, B) := -p(I(y) + rx) - \mu^T_Y(y)q - \frac{1}{2} \text{Tr}(\Sigma_Y \Sigma_Y^T(y)B) - \\
\sup_{\pi \in \Delta_c(y)} \left\{ p\rho(x)\pi(\mu(y) - r\mathbb{1}_N) + \rho(x)\pi \Sigma \Sigma_Y^T(y)g + \frac{1}{2} A\rho^2(x)\|\pi\Sigma(y)\|_2^2 \right\}.
\]
Theorem (Continued)

- Suppose in addition that all $f_k$’s are differentiable, concave and increasing. Then, in the original domain $\tilde{S}_+$:
  (e) $V$ is concave in $x$.
  (f) $\theta^*_k(x, y) = \inf \left\{ \theta \in [0, x/G_k(y)] : \partial_x^+ V(T_k, x - G_k(y)\theta, y) > \frac{\alpha_k}{G_k(y)} f'_k(\theta) \right\} \wedge 1 \wedge \frac{x}{G_k(y)}$.
  (g) Then $\theta^*_k(x, y) = 0$ if $0 \leq x \leq b_k(y)$.
  (h) The consumption threshold $b_k(y) \leq s(T_k, y)$. Moreover, $b_k(y) > 0$ if $\alpha_k f'_k(0)/G_k(y) \leq \max_{j > k} \mathbb{E}[\alpha_j f'_j(0)/G_j(Y_{T_j}^{T_k}, y)]$ and $\mathbb{E}\left[ \int_{T_k}^{T_M} I(Y_s^{T_k}, y) ds \right]$ is sufficiently small.
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Numerical Method

- Finite Difference Explicit Euler Method
- Convergence of numerical solution to viscosity solution guaranteed by Barles and Souganidis [1991], and Forsyth and Labahn (2007)
Market Parameters

- No stochastic factor
- Risk-free rate: $r = 0.05\%$
- Two risky assets:
  - The mean of returns are given by
    \[
    \mu = \begin{pmatrix}
    4.93\% \\
    8.86\%
    \end{pmatrix}
    \]
  - The covariance of returns is given by
    \[
    \Sigma \Sigma^T = \begin{pmatrix}
    0.0017, -0.0021 \\
    -0.0021, 0.0392
    \end{pmatrix}
    \]
  - Drifts and covariance matrix calibrated using historical returns from the 20 year period between January 1998 to December 2017 for index funds representing U.S. bonds and U.S stocks.
Client Parameters

- Investment goals:
  - $G_1 = 30,000, T_1 = 6$ months (buy a car)
  - $G_2 = 10,000, T_2 = 2$ years (vacation)
  - $G_3 = 250,000, T_3 = 5$ years (down payment)

- $I = \$1,000$/month

- $x_0 = \$10,000$

- $\alpha_1 = \alpha_2 = \alpha_3 = \frac{1}{3}$

- Maximum portfolio volatility $c = 7.5$

- Two client profiles:
  - **Flexible goals**: $f_k(\theta) = \theta$
  - **Inflexible goals**: $f_k(\theta) = 1_{\theta=1}$
Value Functions

Figure: Left panel: Value function $V(t, x)$ for a client with linear funding preferences. Right panel: Value function $V(t, x)$ for all-or-nothing funding preferences.
Return and Volatility of Optimal Portfolio: Linear

\[ r + \pi(t, x)(\mu - r1_N) \]

\[ \|\pi(t, x)\Sigma\| \]
Return and Volatility of Optimal Portfolio: All-or-Nothing
Volatility of Optimal Portfolio vs Goal Deadlines

Figure: Top panel: Volatility of the optimal portfolio $\|\pi(t, x)\Sigma\|$ sliced at three different times prior to the first goal deadline. Bottom panel: Volatility of the optimal portfolio sliced at three different times between the first and second goal deadline. Preferences in the funding ratio are linear.
Volatility of Optimal Portfolio: Linear vs All-or-Nothing

Figure: Left panel: Portfolio volatility sliced at time $t = 0.8 T_1$, for all-or-nothing and linear funding ratio preferences. Right panel: Portfolio volatility sliced at time $t = 0.2 T_1 + 0.8 T_2$ (i.e., close to the second goal deadline), for all-or-nothing and linear funding ratio preferences.
Higher Priority of Goal “1” Relative to Goal “2”

Figure: Preferences in the funding ratio are linear.
### Simulated Funding Ratios: Linear

<table>
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<th>( x_0 )</th>
<th>( l )</th>
<th>( \mathbb{E}[\theta_1] )</th>
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Simulated Funding Ratios: All-or-Nothing

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Consumption Boundary
Outline

1. Introduction
2. Sequential Goal-Based Wealth Management (SGBWM)
3. SGBWM for Linear Preferences
4. SGBWM For General Preferences
5. Numerical Analysis
6. Conclusion
Conclusion

- Introduced a continuous time framework for sequential goal-based wealth management

- Consumption and goal fundedness depend on client (goal priorities, amounts, deadlines, income) and market characteristics (drift, volatility)

- Tradeoff between allocating towards current goal v.s. saving for future goals

- Different investment and allocation strategy between a client with flexible and inflexible goals
Desirable Extensions

- Integration of goal-based with lifecycle investing
- **Emergency fund**: require handling goals with random deadlines and random amounts
- More flexible goal characteristics: probability distribution on goal deadlines and goal amounts
- Goals characterized by several subgoals (e.g. pay for children’s education)
Thank you!
## Different Tiers of Investment Goals

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**Table:** Four tiers of goals (Brunel 2015)
Computational Approach I

- Map each portfolio $\pi \in \Delta_c(y)$ to a point in $\mathbb{R}^{2+d}$:

$$\Phi(\pi; y) := (\|\pi\Sigma(y)\|, r + \pi(\mu(y) - r1_N), \pi\Sigma\Sigma^\top_Y(y)).$$

- The image of $\Delta_c(y)$ under $\Phi(\cdot; y)$ is called the opportunity set $P_c(y)$.

- Replacing $\pi \in \Delta_c(y)$ with $(s, m, \nu) \in P_c(y)$ in the HJB equation, we get

$$V_t + I(y)V_x + \mu_Y^\top(y)V_y + xH_c(V_x, V_{xy}, xV_{xx}, y) + \frac{1}{2} \text{Tr}(\Sigma_Y\Sigma^\top_Y(y)V_{yy}) = 0,$$

where

$$H_c(a, g, A, y) = \sup_{(s, m, \nu) \in P_c(y)} \left\{ am + \nu g + \frac{1}{2} As^2 \right\}$$
Computational Approach II

- Rewrite

\[ \mathcal{H}_c(a, g, A, y) = A \inf_{(s, m, \nu) \in \Gamma_c(y)} \left\{ \frac{a}{A} m + \frac{\nu g}{A} + \frac{1}{2} s^2 \right\} =: Ah_c \left( \frac{a}{A}, \frac{g}{A}, y \right), \]

where

\[ \Gamma_c(y) := \{(s_{\min}(m, \nu; y, c), m, \nu)\} \]

is the minimum variance hypersurface

- Assume no factor process
  - \( h_c(\gamma_1) = \inf_{(s, m) \in \Gamma_c} \{\gamma_1 m + \frac{1}{2} s^2\} \).
  - Unique optimizer for \( h_c(\gamma_1) \) resembles tangency portfolio: intersection or tangency of the line with smallest horizontal intercept

- Total complexity of the HJB numerical procedure goes from \( C_N n_t n_x n_y \) down to \( C_N n_y + C_{d+1} n_t n_x n_y \)
Related Literature


- Cvitanic et al. (2020): weighted average of probabilities of achieving target levels and avoiding specified loss levels; single deadline.

- Capponi et al. (2020): personalized robo-advising with dynamic mean-variance risk preferences.
HJB Equation and Minimum Variance Portfolio

- Show that optimizers of the Hamiltonian may be reduced to Markowitz-type minimum variance portfolios.
- Prove that the optimal portfolio lies in a factor-dependent opportunity set defined by the maximum and minimum variance hypersurfaces.
- Develop a geometrical representation of mean and variance of optimal portfolio in terms of an analogue of capital asset line and tangency portfolio.
- Decompose the dynamic optimization problem into two distinct optimization problems, one of which is independent of client characteristics → reduction of computational complexity from $C_N n_t n_x n_y$ to $C_N n_y + C_{d+1} n_t n_x n_y$. 