Systemic Risk Models with Smoothed Endogenous Contagion

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We consider a mathematical finance model for contagion in large financial networks or large portfolios with defaultable entities.

Consider a system with $N$ banks. We say that the $i^{th}$ bank defaults when its assets $A^i_t$ hits a default barrier $D^i_t$. A simple model could be of the form

$$X^i_t = \log(A^i_t) - \log(D^i_t)$$

$$= X^i_{0-} + \int_0^t b(s) \, ds + \sigma (1 - \rho^2)^{1/2} W^i_t + \sigma \rho W^0_t, \quad i = 1, \ldots, N,$$

where $X^i_{0-} \sim i.i.d. \ X^0_{0-}$ and $(W^i_{t})_{t \geq 0}$ are independent standard Brownian motions. This is called the \textit{distance-to-default}. The common noise, $W^0$, is used to model a group of particularly influential banks.
Motivation
A Real World Example

The default of one bank causes the other banks to lose a proportion \( \alpha/N \), \( \alpha > 0 \) constant, of their assets. This is called the contagion. We may impose a continuous notion of latency whereby the impact of contagion is realized gradually as counterparty exposures are sorted out and indirect effects start to kick in. The new assets values are

\[
\hat{A}_{t}^{i,N} := \prod_{j=1}^{N} \left( 1 - \frac{\alpha}{N} \int_{0}^{t} \kappa(t - s) \mathbb{1}_{\{s \geq \tau_{j,N}\}} \, ds \right) A_{t}^{i} \\
\simeq \exp \left( -\alpha \int_{0}^{t} \kappa(t - s) \frac{1}{N} \sum_{j=1}^{N} \mathbb{1}_{\{s \geq \tau_{j,N}\}} \, ds \right) A_{t}^{i}, \quad t < \tau_{t,N}^{i}
\]

where \( \tau_{t,N}^{i} := \inf\{t > 0 : X_{t}^{i,N} \leq 0\} \) for \( X_{t}^{i,N} = \log(\hat{A}_{t}^{i,N}) - \log(D_{t}^{i}) \)
Taking logarithms, the new distance-to-default, $X_{t}^{i,N}$, satisfy

\[
X_{t}^{i,N} = X_{0}^{i} - \int_{0}^{t} b(s) \, ds + \sigma(1 - \rho^{2})^{1/2} W_{t}^{i} + \sigma \rho W_{t}^{0} - \alpha \int_{0}^{t} \kappa(t - s)L_{s}^{N} \, ds
\]

\[
\tau^{i,N} = \inf\{t > 0 \, ; \, X_{t}^{i,N} \leq 0\}
\]

\[
L_{t}^{N} = \frac{1}{N} \sum_{j=1}^{N} \mathbb{1}_{\{t \geq \tau^{j,N}\}}
\]

\[
X_{0}^{i} - \sim X_{0}^{i-}.
\]
Motivation

A Real World Example

(a) Common Noise Path

(b) Density plot of a large system of particles
Motivation

Under suitable assumptions, by Theorem 2.7 from Hambly et al., 2019b, the mean-field limit to (1.1) exists and is unique. The mean-field limit is

\[
\begin{cases}
X^\kappa_t = X_0^- + \int_0^t b(s) \, ds + \sigma (1 - \rho^2)^{1/2} W_t \\
\quad + \sigma \rho W_0^0 - \alpha \int_0^t \kappa(t - s) L_s^\kappa \, ds \\
\tau^\kappa = \inf \{ t > 0 ; X^\kappa_t \leq 0 \} \\
L_t^\kappa = \mathbb{P} [ \tau^\kappa \leq t | W^0 ] \\
\mathcal{L}_t^\kappa = \int_0^t \kappa(t - s) L_s^\kappa \, ds,
\end{cases}
\]

where \( \kappa \in \mathcal{W}_0^{1,1} (\mathbb{R}_+) \), \( \kappa \geq 0 \), and \( \| \kappa \|_1 = 1 \).
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Main Results

Now fix a mollifier $\kappa$ and set $\kappa^\varepsilon := \varepsilon^{-1}\kappa(\frac{\cdot}{\varepsilon})$ for $\varepsilon > 0$. Then we consider the system of equations

\[
\begin{align*}
X_t^\varepsilon &= X_0^- + \int_0^t b(s) \, ds + \sigma(1 - \rho^2)^{1/2} W_t \\
&\quad + \sigma \rho W_t^0 - \alpha \int_0^t \kappa^\varepsilon(t - s)L_s^\varepsilon \, ds \\
\tau^\varepsilon &= \inf\{t > 0 ; X_t^\varepsilon \leq 0\} \\
L_t^\varepsilon &= \mathbb{P}[\tau^\varepsilon \leq t | W^0] \\
\mathcal{L}_t^\varepsilon &= \int_0^t \kappa^\varepsilon(t - s)L_s^\varepsilon \, ds
\end{align*}
\]
Main Results

(a) Loss

(b) Rate of Convergence

**Figure:** The rate of convergence in the simplified setting $X_t = X_0^- + W_t - \alpha \mathcal{L}_t^\varepsilon$ with $X_0^- \sim^d \Gamma(1.5, 2)$, $\alpha = 1.3$
Main Results

Intuitively, as $\epsilon \rightarrow 0$ we expect convergence to a system with instantaneous loss

$$\begin{cases} X_t = X_0^- + \int_0^t b(s) \, ds + \sigma (1 - \rho^2)^{1/2} W_t \\ + \sigma \rho W_t^0 - \alpha \mathbb{P} \left[ \tau \leq t \mid W^0 \right] \\ \tau = \inf \{ t > 0 ; X_t \leq 0 \} \\ L_t = \mathbb{P} \left[ \tau \leq t \mid W^0 \right] \end{cases}$$

(2.2)

If we establish this convergence, as a side result we get the existence of physical solutions in a case that hasn’t been studied before.
Motivation

A Real World Example

(a) Common Noise Path

(b) Density plot of a large system of particles
Main Results

The literature is rich with results pertaining to equations of the form

\[ X_t = X_{0-} + \int_0^t b(s) \, ds + \sigma (1 - \rho^2)^{1/2} W_t + \sigma \rho W^0_t - \alpha \mathbb{P} \left[ \inf_{s \leq t} X_s \leq 0 \mid W^0 \right] \]

where \( W^0 \) and \( W \) are independent standard Brownian Motions and \( \alpha > 0 \) is constant.

- Existence of relaxed solutions shown in (Ledger et al., 2021)
- \((\sigma = 1, \rho = 0)\) Existence of strong solutions shown in (Cuchiero et al., 2020; Hambly et al., 2019a; Delarue et al., 2015)
- Regularity properties are shown in (Ledger et al., 2020; Delarue et al., 2022; Hambly et al., 2019a) etc.
By eq. (2.2), the jumps of $L$ must satisfy the equality

$$
\Delta L_t = \mathbb{P}[X_{t^-} - \alpha \Delta L_t \leq 0, \tau \geq t \mid W^0] \quad \text{a.s.}
$$
By eq. (2.2), the jumps of $L$ must satisfy the equality

$$\Delta L_t = \mathbb{P}[X_{t^-} - \alpha \Delta L_t \leq 0, \tau \geq t \mid W^0] \quad \text{a.s.}$$

This equation is not sufficient to classify the jumps of $L$. We are interested in finding solutions that will have minimal jumps. That is $L$ will satisfy the minimal jump condition

$$\Delta L_t = \inf\{z > 0 ; \mathbb{P}[X_{t^-} - \alpha z \leq 0, \tau \geq t \mid W^0] < z\}.$$
Theorem (Existence and convergence)

Given suitable assumptions, there exists a solution to eq. (2.2). Furthermore, we have that \( X^\varepsilon \) converges almost surely to \( X \) in \((D_\mathbb{R}, M1)\), the space of cádlág functions endowed with the M1 topology. Lastly, we may characterise the size of the jumps of \( L \) by the minimal jump condition

\[
\Delta L_t = \inf \{ z > 0 ; \mathbb{P}[X_{t^-} - \alpha z \leq 0, \tau \geq t | W^0] < z \} \quad \text{a.s.}
\]
Main Results

Theorem (Existence and convergence)

Given suitable assumptions, there exists a solution to eq. (2.2). Furthermore, we have that $X^\varepsilon$ converges almost surely to $X$ in $(D_\mathbb{R}, M1)$, the space of cádlág functions endowed with the M1 topology. Lastly, we may characterise the size of the jumps of $L$ by the minimal jump condition

$$\Delta L_t = \inf\{z > 0; \mathbb{P}[X_{t^-} - \alpha z \leq 0, \tau \geq t | W^0] < z\} \text{ a.s.}$$

Proof.
Employ a stochastic fixed point argument and continuity of hitting times.
Main Results

Proof.
First, we construct a fixed point to the map $\Gamma$ in
\[
\begin{align*}
\dot{X}_t^\ell &= b(t) \, dt + \sigma \sqrt{1 - \rho^2} \, dW_t + \sigma \rho \, dW_t^0 - \alpha \, d\ell_t, \\
\tau^\ell &= \inf\{t > 0 : X_t^\ell \leq 0\}, \\
[\ell]_t &= P\left[\tau^\ell \leq t \mid \mathcal{F}_t^{W^0}\right],
\end{align*}
\]
where $\ell$ is a $W^0$-measurable cádlág process. Call this fixed point $\bar{L}$.
Main Results

Proof.
First, we construct a fixed point to the map $\Gamma$ in

$$
\begin{aligned}
\left\{\begin{array}{l}
dX^\ell_t &= b(t) \, dt + \sigma \sqrt{1 - \rho^2} \, dW_t + \sigma \rho \, dW^0_t - \alpha \, d\ell_t, \\
\tau^\ell &= \inf\{t > 0 ; \, X^\ell_t \leq 0\}, \\
\Gamma[\ell]_t &= P \left[ \tau^\ell \leq t \mid \mathcal{F}^W_t \right],
\end{array}\right.
\end{aligned}
$$

where $\ell$ is a $W^0$-measurable cádlág process. Call this fixed point $\bar{L}$.
Similarly, we construct a fixed point for the smoothed system and denote it by $L^\varepsilon$. 
Main Results

Proof.
First, we construct a fixed point to the map $\Gamma$ in

\[
\begin{cases}
    dX^\ell_t = b(t) \, dt + \sigma \sqrt{1 - \rho^2} \, dW_t + \sigma \rho \, dW_t^0 - \alpha \, d\ell_t, \\
    \tau^\ell = \inf\{t > 0 \mid X^\ell_t \leq 0\}, \\
    \Gamma[\ell]_t = \mathbb{P}\left[\tau^\ell \leq t \mid \mathcal{F}^W_t\right],
\end{cases}
\]

where $\ell$ is a $W^0$-measurable cádlág process. Call this fixed point $\bar{\ell}$. Similarly, we construct a fixed point for the smoothed system and denote it by $\bar{\ell}^\varepsilon$.

Lastly, we show along any countable subsequence, $\bar{\ell}^\varepsilon$ has an almost sure limit point and this limit point must be $\bar{\ell}$.  

\[\square\]
In fact, the theory in Hambly et al., 2019b allows us to generalise the coefficients of eq. (2.1) and consider the system of equations

\[
dX_t^\varepsilon = b(t, X_t^\varepsilon, \nu_t^\varepsilon) \, dt + \sigma(t, X_t^\varepsilon) \sqrt{1 - \rho(t, \nu_t^\varepsilon)^2} \, dW_t
+ \sigma(t) \rho(t, \nu_t^\varepsilon) \, dW_0^t - \alpha(t) \, d\mathcal{L}_t^\varepsilon,
\]

\[
\tau^\varepsilon = \inf\{t > 0 \mid X_t^\varepsilon \leq 0\},
\]

\[
P^\varepsilon = \mathbb{P} \left[ X^\varepsilon \in \cdot \mid W^0 \right], \quad \nu_t^\varepsilon := \mathbb{P} \left[ X_t^\varepsilon \in \cdot, \tau^\varepsilon > t \mid W^0 \right],
\]

\[
L_t^\varepsilon = P^\varepsilon \left[ \tau^\varepsilon \leq t \right], \quad \mathcal{L}_t^\varepsilon = \int_0^t \kappa^\varepsilon(t - s) L_s^\varepsilon \, ds.
\]
Main Results

Theorem (Existence and convergence generalised)

Given suitable assumptions, we may construct a solution equation

\[
\begin{align*}
\frac{dX_t}{dt} &= b(t, X_t, \nu_t) dt + \sigma(t, X_t) \sqrt{1 - \rho(t, \nu_t)^2} dW_t \\
&\quad + \sigma(t, X_t) \rho(t, \nu_t) dW^0_t - \alpha(t) dL_t,
\end{align*}
\]

\[\tau = \inf\{t > 0 ; X_t \leq 0\},\]

\[P = P[ X \in \cdot | W^0, P ], \quad \nu_t := P[ X_t \in \cdot, \tau > t | W^0, P ],\]

\[L_t = P[ \tau \leq t ],\]

\begin{equation}
(2.4)
\end{equation}

such that for a suitable subsequence \((\epsilon_n)_{n \geq 1}\), we have

\[\text{Law}(\tilde{X}^{\epsilon_n} | W^0) \rightarrow \text{Law}(\tilde{X} | P, W^0),\]

where \(\tilde{X}^{\epsilon_n}\) and \(\tilde{X}\) are extensions of \(X^{\epsilon}\) and \(X\). Lastly, we have an upper bound on the jumps of \(L\),

\[\Delta L_t \leq \inf \{ x \geq 0 : \nu_t - [0, \alpha(t)x] < x \}.\]
Main Results

Proof Outline.

(i) Tightness of \( \{ \text{Law}(\tilde{X}^\varepsilon \mid W^0) \}_{\varepsilon > 0} \subset \mathcal{P}(D([-1, T + 1], \mathbb{R})) \).

(ii) Convergence of delayed loss to loss in limiting system

(iii) Martingale arguments

(iv) Estimates on increments of limiting loss
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Rates of Convergence

Continuous Setting

Proposition

Let \((X_t, L_t)_{t \geq 0}\) be a physical solution to \(X_t = X_0^- + W_t - \alpha L_t\). Suppose that \(L \in C^1([0, +\infty))\), then for all \(t_0 > 0\) there exists a constant \(\tilde{K} = \tilde{K}(t_0)\) s.t.

\[
\sup_{s \in [0, t_0]} |L_s - L_s^\epsilon| \leq \tilde{K} \epsilon^{1/2}
\]

Proof.

Upper bound the difference between \(L\) and \(L^\epsilon\) by an integral of the differences then apply a generalised Gronwall inequality.
Proposition

Let \((X_t, L_t)_{t \geq 0}\) be a physical solution to \(X_t = X_0 - \alpha L_t + W_t\). Suppose that \(L \in C^1([0, +\infty))\), then for all \(t_0 > 0\) there exists a constant \(\tilde{K} = \tilde{K}(t_0)\) s.t.

\[
\sup_{s \in [0, t_0]} |L_s - L^\epsilon_s| \leq \tilde{K}\epsilon^{1/2}
\]

Proof.

Upper bound the difference between \(L\) and \(L^\epsilon\) by an integral of the differences then apply a generalised Gronwall inequality.
Rates of Convergence
Continuous Setting

Figure: The rate of convergence in the simplified setting $X_t = X_0 - W_t - \alpha L_t$
with $X_0 \sim^d \text{Uniform}[0.45, 0.55]$, $\alpha = 0.9$
Proposition

Let \((X_t, L_t)_{t \geq 0}\) be a physical solution to \(X_t = X_0^- + W_t - \alpha L_t\). Given ‘suitable assumptions’ on the initial condition, there exits a \(t_{\text{explode}} > 0\) such that for any \(t_0 \in (0, t_{\text{explode}})\) such that

\[
\sup_{s \in [0, t_0]} |L_s - L^c_s| \leq \tilde{K} \epsilon^{\beta/2},
\]

where \(\tilde{K} = \tilde{K}(t_0)\) is a constant and \(\beta \in (0, 1)\) depends on \(X_0^-\).
Rates of Convergence
Continuous Setting

(a) Loss

(b) Rate of Convergence

Figure: The rate of convergence in the simplified setting $X_t = X_{0-} + W_t - \alpha L_t$ with $X_{0-} \sim^d \Gamma(1.2, 0.5)$, $\alpha = 0.9$
Concluding Remarks

(i) We provide a novel method via weak convergence to prove the existence of relaxed solutions to a McKean-Vlasov equation with positive feedback and endogenous contagion in a setting that has not been studied before.

(ii) Provided the coefficients depend only on time, have shown the existence of $W^0$—measureable physical solution and have almost sure convergence of the delayed system to the singular system.

(iii) We provide explicit rates of convergence prior to the first time regularity of the loss process decays.


