Weak Convergence of Stochastic Integrals on Skorokhod Space: the J1 & M1 Topologies

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Stochastic Control & Financial Engineering
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The type of question we are interested in

- $D_{\mathbb{R}^d}[0, \infty)$ endowed with Skorokhod’s J1 or M1 topology
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  $X^n \Rightarrow X$ on $(D_{\mathbb{R}^d}[0, \infty), \rho)$, for $\rho \in \{d_{M1}, d_{J1}\}$
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When can we have a form of ‘weak continuity’ of Itô integrals:

$$(X^n, \int_0^\bullet H^n_{s-} \, dX^n_s) \Rightarrow (X, \int_0^\bullet H_{s-} \, dX_s) \text{ on } (D_{\mathbb{R}^{2d}}[0, \infty), \rho)$$

whenever $H^n \Rightarrow H$ on $(D_{\mathbb{R}^d}[0, \infty), \tilde{\rho})$ for $\tilde{\rho} \in \{d_{M1}, d_{J1}\}$
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- Clearly want $(H^n, X^n) \Rightarrow (H, X)$ on $(D_{\mathbb{R}^d}[0, \infty), \rho) \times (D_{\mathbb{R}^d}[0, \infty), \tilde{\rho})$
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- J1 case: Kurtz–Protter (’91), Jakubowski–Memin–Pages (’89)
  - $\rightsquigarrow$ Control on $X^n$: UCV or P-UT condition (in given filtration)
  - $\rightsquigarrow$ Interplay of $(H^n, X^n)$: ‘converge together’ on $(D_{\mathbb{R}^{2d}}[0,\infty), d_{J1})$
Motivation and the classical theory
New results on a J1-&-M1 framework
More on applications

The classical J1 topology

• Idea of the J1 topology: two càdlàg processes should be close to each other if:
  we can do a minor perturbation of the timing of jumps to make their paths uniformly close with the perturbation being uniformly close to the identity.

Definition of J1 metric
\[ d_{J1}(x_1, x_2) := \inf_{\lambda \in \Lambda} \left\{ |\lambda - \text{Id}| \vee |x_1 \circ \lambda - x_2| \right\} \]
\[ \|x\|_{J1} := \sup_{t \in [0, T]} |x(t)| \] is the uniform norm
\[ \Lambda \] is the set of increasing homeomorphisms \[ \mathbb{R} \to \mathbb{R} \]
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The M1 topology

- Given \( x \in D_{\mathbb{R}^k}[0, T] \), define its \textit{completed graph} as the set

\[
\Gamma_x := \{(z, t) \in \mathbb{R}^k \times [0, T] : z = \alpha x(t-) + (1 - \alpha)x(t), \ \alpha \in [0, 1]\}
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- Put \textit{total order} \( \preceq \) on \( \Gamma_x \) given by:

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(z_1, t_1) \preceq (z_2, t_2) \iff t_1 < t_2 \ \text{or} \ (t_1 = t_2 \ \text{and} \ |x^{(i)}(t_1) - z_1^{(i)}| \leq |x^{(i)}(t_1) - z_2^{(i)}| \ \forall i)
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- Put total order `$\leq$' on $\Gamma_x$ given by: $(z_1, t_1) \leq (z_2, t_2) \iff t_1 < t_2 \text{ or } (t_1 = t_2 \text{ and } |x^{(i)}(t_1) - z_1^{(i)}| \leq |x^{(i)}(t_1) - z_2^{(i)}| \forall i)$

- *Parametric representations*: cont., non-decreasing, surjective $(u, r):[0, 1] \rightarrow \Gamma_x$. Let $\Pi(x)$ be set of all parametric reps of $\Gamma_x$. 
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\[ d_{M1}(x_1, x_2) := \inf_{(u_j, r_j) \in \Pi(x_j) \atop j=1,2} (\ |u_1 - u_2|_1^* \lor |r_1 - r_2|_1^* ) \]
Suppose asset price $X_t$ is determined by price moves due to trades happening at unevenly-spaced discrete times.
Microstructural CTRW models of asset prices

- Suppose **asset price** $X_t$ is determined by **price moves** due to **trades** happening at **unevenly-spaced discrete times**.

  $\sim \quad$ Non-Exponential **waiting times** $\sigma_i := t_i - t_{i-1}$

  satisfying $\mathbb{P}(\sigma_i > t) \sim t^{-\beta}$, $\beta \in (0, 1)$, $\beta$ close to 1

  $\sim \quad$ Non-Gaussian **log price innovations** $\xi_i := \ln(X_{t_i}) - \ln(X_{t_{i-1}})$

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Summing up the price innovations, we obtain a CTRW

$$X_t = e^{R_t}, \quad R_t := \sum_{i=0}^{N_t} \xi_i, \quad N_t := \max\left\{k \geq 1 : \sum_{i=1}^{k} \sigma_i \leq t\right\}$$
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Higher-frequency trading: lots of focus on trade durations obeying power laws (sub-diffusive, high activity vs. constancy)
Scaling limits of CTRWs

- Let $Z$ denote a symmetric $\alpha$-stable Levy process
- Let $D$ denote a $\beta$-stable subordinator
- Define the ‘inverse’ subordinator $D_t^{-1} := \inf\{s \geq 0 : D_s > t\}$
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- Jacquier & Torricelli (2020) build a nice option pricing theory for such limiting models $X_t = e^{rt + R_t}$ (with tempering)
Scaling limits of CTRWs: J1 versus M1

- As before, consider i.i.d. waiting times $\sigma_i$ and let

$$N_t := \max\left\{ k \geq 1 : \sum_{i=1}^{k} \sigma_i \leq t \right\}$$

- Asset price $X^n = e^{R^n}$ with autoregressive price innovations

**CTRW with correlated jump sizes**

$$R_t^n := \sum_{i=0}^{N_{nt}} \frac{\xi_i}{n^{\beta/\alpha}}, \quad \xi_i = \sum_{k=0}^{K} c_k \zeta_{i-k} \quad (c_k \geq 0; \ \zeta_i \ \text{i.i.d.}; \ \zeta \perp \sigma)$$

- Scaling limits studied by Meerschaert, Nane, Xiao (’09)

**Weak convergence to subordinated $\alpha$-stable Levy process**

$$R_t^n \implies R_t := \left( \sum_{k=0}^{K} c_k \right) Z_{D_t} \quad \text{in M1}$$
Stability of the financial gain?

- When doing pricing in a model that arises as a scaling limit of CTRWs, one is naturally led to consider the stability of the financial gain process

\[ \int_0^t H_{s-} \, dX_s \]

- A general left-continuous trading strategy \( H_{-} \) could e.g. be of the form

\[ H_{s-} = f(s, X_{s-}) \]

- Such a trading strategy \( H_{-} \) should be representative of a family of simple predictable trading strategies \( H^n_{-} \) with respect to the CTRW asset prices \( X^n \) such that

\[ \int_0^t H^n_{-} \, dX^n \quad \Rightarrow \quad \int_0^t H \, dX \]
Let $X^n$ be a sequence of semimartingales satisfying:

**Predictable-Uniform Tightness (P-UT) condition**

For all $t \geq 0$, the family of random variables

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\mathcal{A}_t := \bigcup_{n \in \mathbb{N}} \left\{ \int_0^t G_s \, dX^n_s : G \in S(F^n), \ |G|^*_t \leq 1 \right\}
$$

is **tight** in $\mathbb{R}$, where $S(F^n)$ denotes the space of simple predictable processes on the given filtered probability space $(\Omega^n, \mathcal{F}^n, F^n, \mathbb{P}^n)$. 
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If $(H^n, X^n) \Rightarrow (H, X)$ on $(D_{\mathbb{R}^{2d}}[0, \infty), J1)$, $\mathbb{F}^n$-adapted, then

$$\left( H^n, X^n, \int_0^\cdot H^n_{s-} \, dX^n_s \right) \Rightarrow \left( H, X, \int_0^\cdot H^n_{s-} \, dX_s \right) \text{ on } (D_{\mathbb{R}^{3d}}[0, \infty), J1)$$
The J1 theory of Kurtz & Protter

- Consider a sequence of semimartingales $X^n$

- Define $J_\delta(X^n) := \sum_{0 < s \leq t} \left( \left( 1 - \frac{\delta}{|\Delta X^n_s|} \right) \lor 0 \right) \Delta X^n_s$

- And set $X^{n,\delta} := X^n - J_\delta(X^n)$
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Uniformly Controlled Variations (Kurtz & Protter, 1991)

\( (X_n)_{n \geq 1} \) has UCV property for \( \delta > 0 \) if \( X^{n,\delta} = M^{n,\delta} + A^{n,\delta} \) s.t.:

for each \( \alpha > 1 \) there exist \( \mathbb{F}_n \)-stopping times \( \tau^n_{\alpha} \) satisfying

\[
\sup_{n \geq 1} \mathbb{P}^n \left( \tau^n_{\alpha,\delta} \leq \alpha \right) \leq \alpha^{-1} \quad \text{and}
\]

\[
\sup_{n \geq 1} \mathbb{E}^n \left[ [M^{n,\delta}]_{T \wedge \tau^n_{\alpha,\delta}} + TV_{[0, T \wedge \tau^n_{\alpha,\delta}]}(A^{n,\delta}) \right] < \infty \quad \forall T > 0
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- If $(H^n, X^n) \Rightarrow (H, X)$ on $(D_{\mathbb{R}^{2d}[0, \infty)}, J1)$, $\mathbb{F}^n$-adapted, then
  the stochastic integrals converge weakly as on previous slide
The J1 theory of Kurtz & Protter – cont’d

- Arguments are very clean, relying on **three operations** that are **J1 continuous** (a.s.)

- **First map**: integration of simple paths

\[
(x^n, y^n) \mapsto \left( x^n, y^n, \int_0^\bullet x^n(s-)dy^n(s), \int_0^\bullet y^n(s-)dx^n(s) \right)
\]

for \( x^n \) suitable ‘simple’ càdlàg paths with a uniform bound on the number of jumps of the \( x^n \)

- **Second map**: extraction of large jumps through the map \( x^n \mapsto J_\delta(x^n) \) from the UCV condition on previous slide

- **Third map**: a.s. discretization of the integrands through a final map \( x^n \mapsto l_\varepsilon(x^n) \) with uniform \( \varepsilon \) precision
First main result: J1-\&-M1 weak convergence theory

**Definition: good decompositions**

\[ X^n = M^n + A^n, \text{ where } M^n \text{ is a local martingale, } A^n \text{ finite variation.} \]

For each \( t > 0 \) and all \( M^n \)-stopping times \( \tau^n \):

\[
\lim_{R \to \infty} \sup_{n \geq 1} \mathbb{P}^n \left( \text{TV}_{[0,t]}(A^n) > R \right) = 0 \quad \text{and} \quad \sup_{n \geq 1} \mathbb{E}^n \left[ |\Delta M^n_{t \wedge \tau^n}| \right] < \infty
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Theorem: ‘weak continuity’ of stochastic integrals

Let \((H^n, X^n) \Rightarrow (H, X)\) in \((D_{\mathbb{R}^d}, d_{M1}) \times (D_{\mathbb{R}^d}, \rho), \rho \in \{d_{M1}, d_{J1}\}\).

If (i) the \(X^n\) have good decompositions, (ii) the pairs \((H^n, X^n)\) are \(\mathbb{F}^n\)-adapted, and (iii) \((H^n, X^n)\) satisfy AVCO, then

\[
\left( X^n, \int_0^\cdot H^n_s^- \, dX^n_s \right) \Rightarrow \left( X, \int_0^\cdot H_s^- \, dX_s \right) \quad \text{in} \quad (D_{\mathbb{R}^{2d}}, \rho).
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AVCO = asymptotically vanishing consecutive oscillations (of the \(H^n\) before the \(X^n\)). Comes from Jakubowski ('96).
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A general J1-\&-M1 theory of weak convergence

Theorem: ‘weak continuity’ of stochastic integrals

Let \((H^n, X^n) \Rightarrow (H, X)\) in \((D_{\mathbb{R}^d}, dM_1) \times (D_{\mathbb{R}^d}, \rho)\), \(\rho \in \{dM_1, dJ_1\}\).

If (i) \((H^n, X^n)\) \(\mathbb{F}^n\)-adapted, (ii) the \(X^n\) have good decompositions, and (iii) \((H^n, X^n)\) satisfy AVCO, then

\[
\left( X^n, \int_0^t H^n_s \, dX^n_s \right) \Rightarrow \left( X, \int_0^t H_s \, dX_s \right) \quad \text{in} \quad (D_{\mathbb{R}^{2d}}, \rho).
\]

\[\text{AVCO = asymptotically vanishing consecutive oscillations (of the } H^n \text{ before the } X^n). \text{ Comes from Jakubowski ('96).}\]

Proposition: two fundamental sufficient criteria

AVCO is satisfied if one of the following holds:

1. The sequence \((H^n, X^n)\) converges weakly to \((H, X)\) in \((D_{\mathbb{R}^{2d}}, dJ_1)\).
2. The limiting processes \(H\) and \(X\) satisfy
   \[\text{Disc}(H) \cap \text{Disc}(X) = \emptyset\quad \text{almost surely.}\]
The AVCO condition

- For $\delta > 0$ we define the function of the largest consecutive oscillations $\hat{w}_\delta : D_{\mathbb{R}^d}[0, T] \times D_{\mathbb{R}^d}[0, T] \rightarrow \mathbb{R}_+$ by

$$\hat{w}_\delta(h, x) := \sup\{|h(s) - h(t)| \land |x(t) - x(u)| : s < t < u \leq (s + \delta) \land T\}$$

Asymptotically Vanishing Consecutive Oscillations (AVCO)

We say the sequence $(H^n, X^n)_{n \geq 1}$ satisfies the AVCO condition if for every $\gamma > 0$ and $T > 0$, it holds that

$$\lim_{\lambda \downarrow 0} \limsup_{n \to \infty} \mathbb{P}^n \left( \hat{w}_\lambda(H^n|_{[0, T]}, X^n|_{[0, T]}) > \gamma \right) = 0$$

where $H^n|_{[0, T]}$, $X^n|_{[0, T]}$ are the restrictions of $H$ and $X$ to $[0, T]$.

- This is all borrowed from Jakubowski (AOP '96)
The AVCO condition – an illustrative example

\[ x_n := \frac{1}{2} \mathbf{1}_{\left[1 - \frac{2}{n}, 1\right]} + \mathbf{1}_{[1, 2]}, \quad y_n := \frac{1}{2} \mathbf{1}_{\left[1 - \frac{1}{n}, 1\right]} + \mathbf{1}_{[1, 2]}, \quad x = y = \mathbf{1}_{[1, 2]} \]
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\[ \int_0^2 x_n(s-) \, dy_n(s) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2} \]
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\[ \int_{0}^{2} x(s^-) \, dy(s) = 0 \]
Some confusion in the econometrics literature

- Fabozzi, Paulauskas, and Rachev (Econometric Theory, 2011)
  
  Thm. 2 in Caner (1997), which is subsequently used in Thms. 3, 4, and 5 and then applied in Caner (1998), is not proven.
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  - “the principal purpose of this comment, in addition to pointing out the gap in the proof of Thm. 2 in Caner (1997), is to **warn researchers working in econometrics about the difficulties associated with the convergence problems for stochastic integrals and the possible misuse of the continuous mapping theorem**”
Some confusion in the econometrics literature - cont’d

- Caner (Econometric Theory, 1997; J. of Econometrics, 1998)

Start of first paper, states and proves the following theorem:

THEOREM 2. For each $n$, let $(X_n, P_n)$ be an $\{\mathcal{F}_{nt}\}$–adapted process with sample paths in $D_{R^{dk}\times R^k}[0,1]$, and let $P_n$ be an $\{\mathcal{F}_{nt}\}$ semimartingale. Suppose that $P_n = P_n^{(1)} + P_n^{(2)}$, where $P_n^{(2)}$ is an $\{\mathcal{F}_{nt}\}$ locally square-integrable martingale and $P_n^{(1)}$ is a constant function except for finitely many discontinuities in any finite time interval. Let $N_n(r)$ denote the number of discontinuities of $P_n^{(1)}$ in the interval $[0, r]$. Suppose $\{N_n(r)\}$ is stochastically bounded for each $r > 0$ and if

$$(X_n, P_n, P_n^{(1)}) \Rightarrow (X, Y, Y^-)$$

in $D_{R^{dk}\times R^k\times R^k}[0,1]$ then $Y$ is a semimartingale with respect to a filtration to which $X$ and $Y$ are adapted and

$$(X_n, P_n, \int X_n dP_n) \Rightarrow (X, Y, \int X^- dY)$$

in $D_{R^{dk}\times R^k\times R^k}[0,1]$, where $X_n$ is a real valued matrix and $X^-(.)$ is the left limit of $X(.)$. 

A counterexample for martingale integrators

In case of local martingale integrators $X^n = M^n$, there is a nice sufficient condition for $\text{P-UT/UCV}$ (due to Jacod):

$$
\sup_{n \geq 1} \mathbb{E} \left[ \sup_{s \in [0, t]} |\Delta M^n_s| \right] < \infty \quad \text{for all} \quad t \geq 0.
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A counterexample for martingale integrators

- In case of local martingale integrators $X^n = M^n$, there is a nice **sufficient condition** for $\mathbb{P}$-$\text{UT}/\text{UCV}$ (due to Jacod):

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- The theorem on previous slide was concerned with $X^n = Z^n + M^n$, where $Z^n$ is nice and $M^n$ is a locally square-integrable local martingale.
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- The theorem on previous slide was concerned with $X^n = Z^n + M^n$, where $Z^n$ is nice and $M^n$ is a locally square-integrable local martingale.

- We construct a martingale $M^n$ such that:
  1. $M^n \to 0$ uniformly a.s.
  2. For all $t \geq 0$, we have $\mathbb{E} \left[ \sup_{s \in [0, t]} |\Delta M^n_s| \right] < \infty$ for all $n \geq 1$
  3. For some $T > 0$, we have $\sup_{n \geq 1} \mathbb{E} \left[ \sup_{s \in [0, T]} |\Delta M^n_t| \right] = \infty$
  4. $M^n$ does not have good decompositions
On M1 versus J1: the case of martingales

- Good decompositions of strictly M1 convergent semimartingale sequences can be tricky
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**Theorem: J1 versus M1 convergence**

Let \( \mathcal{A} := \{ X^i : i \in I \} \) be a family of \( \mathbb{R}^d \)-valued, continuous-time martingales on filtered probability spaces \((\Omega^i, \mathcal{F}^i, \mathbb{F}^i, \mathbb{P}^i)\).

If \( \mathcal{A} \) is weakly relatively compact for the M1 topology, then it is also weakly relatively compact for J1.
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If \( \mathcal{A} \) is weakly relatively compact for the M1 topology, then it is also weakly relatively compact for J1.

- In particular, one obtains the following:

**Corollary: M1-convergent local martingales**

Let \( (X^n)_{n \geq 1} \) be a sequence of local martingales on \((\Omega^n, \mathcal{F}^n, \mathcal{F}^n, \mathbb{P}^n)\) converging weakly on \((D_{\mathbb{R}^d}[0, \infty), d_{M1})\) to some càdlàg limit \( X \).

Then, \( X^n \Rightarrow X \) on \((D_{\mathbb{R}^d}[0, \infty), d_{J1})\).
Consider a **single-delay moving average** of the form

\[ X^n_t := \frac{1}{n^{1-\alpha}} \sum_{k=1}^{\lfloor nt \rfloor} (c_0 Z_k + c_1 Z_{k-1}) \]

with \( c_0, c_1 > 0 \) and i.i.d. random variables \( Z_k \) in the normal domain of attraction of an \( \alpha \)-stable law with \( 1 < \alpha < 2 \).
Consider a single-delay moving average of the form

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Proposition: exploding stochastic integrals

One can find pure jump càdlàg integrands \( H^n \) adapted to the natural filtration of the above \( X^n \) such that:

(i) they converge uniformly a.s. to the zero process, yet

(ii) all finite-dim distributions of \( \int_0^\cdot H^n_s \, dX^n_s \) explode as \( n \to \infty \).
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In particular, \( (X^n)_{n \geq 1} \) does not admit good decompositions.
Co-integrated time series (Paulauskas & Rachev ’98)

- Co-integration with infinite variance innovations $u_k$ and $v_k$

Time series (Park & Phillips, 1988)

$$Y_k = AX_k + u_k \text{ (in } \mathbb{R}^p), \quad X_k = X_{k-1} + v_k \text{ (in } \mathbb{R}^q)$$
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- Regression yields an estimator \( \hat{A} \) of the true \( p \times q \) matrix \( A \)

Least squares estimator

\[
\hat{A} = Y^TX(X^TX)^{-1}, \quad \hat{A} - A = U^TX(X^TX)^{-1}
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**Least squares estimator**

$$\hat{A} = Y^TX(X^TX)^{-1}, \quad \hat{A} - A = U^TX(X^TX)^{-1}$$

- Let $T = \text{diag}(\frac{1}{n^{1/\alpha_1}}, \ldots, \frac{1}{n^{1/\alpha_p}})$ and $\tilde{T} = \text{diag}(\frac{1}{n^{1/\tilde{\alpha}_1}}, \ldots, \frac{1}{n^{1/\tilde{\alpha}_q}})$

**Interested in asymptotics as $n \to \infty$**

$$nT(\hat{A} - A)\tilde{T}^{-1} = (TU^TX\tilde{T})(\frac{1}{n}\tilde{T}X^TX\tilde{T})^{-1}$$
Asymptotic analysis

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\[
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\[ nT(\widehat{A} - A)^\widetilde{T}^{-1} = (TU^TX\widetilde{T})\left(\frac{1}{n} \widetilde{TX}^T\widetilde{X}T\right)^{-1} \]

- For \( t \in [0, 1] \), let

\[ Z_t^{n,i} := \sum_{l=1}^{[nt]} \frac{1}{n^{1/\alpha_i}} u^l_j, \quad \tilde{Z}_t^{n,j} := \sum_{l=1}^{[nt]} \frac{1}{n^{1/\tilde{\alpha}_j}} v^l_j \]

- Rewriting \( TU^TX\widetilde{T} \) leads to the **stochastic integrals**

\[ \int_0^1 Z_t^{n,i} d\tilde{Z}_t^{n,j} \quad \text{for } i = 1, \ldots, p, \ j = 1, \ldots, q. \]
Structure of the innovations

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- ~\( J_1\)-convergence \((Z_n, \tilde{Z}_n) \Rightarrow (Z, \tilde{Z})\) to \(\alpha\)-stable Lévy processes
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- A lot of interest in linear processes for the innovations:
Motivation and the classical theory
New results on a J1- & M1 framework
More on applications

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- A lot of interest in **linear processes for the innovations**:
  \[ \Rightarrow \] Instead of \( (u_k, v_k) \) i.i.d., consider ‘lagged’ dependence

\[
  u_k = \sum_{h=0}^{K} c_h \varepsilon_{k-h}, \quad v_k = \sum_{h=0}^{N} \tilde{c}_h \tilde{\varepsilon}_{k-h} \quad (\varepsilon, \tilde{\varepsilon} \text{ i.i.d.})
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  \[ \Rightarrow \text{If } c_h, \tilde{c}_h \geq 0, \text{ we get M1-convergence} \ (Avram & Taqqu ’91): \]

  \[ Z^n \Rightarrow cZ, \quad \tilde{Z}^n \Rightarrow \tilde{c}\tilde{Z}, \]

  where \( c := \sum_{j=0}^{K} c_j \) and \( \tilde{c} := \sum_{j=0}^{N} \tilde{c}_j \)
Application in insurance: ruin theory with investment

- **Ruin theory with investment**: insurance part (income $P$) independent of external investments (price $S$) — studied by Norberg, Kalashnikov, Paulsen, Gjessing, Steffensen, etc.

- **Stream of income** (inflows less outflows) $P_t$

- Continuously investing its total reserves in an external asset with price process $S_t$

- Norberg & Steffensen explain how total reserves $U$ evolve as

\[
U_{t_i} = U_{t_{i-1}} + U_{t_{i-1}} \frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}} + (P_{t_i} - P_{t_{i-1}})
\]

\[
\implies dU_t = U_t \frac{dS_t}{S_t} + dP_t.
\]
Application in insurance: ruin theory with investment

- **Total reserves** $U_t$ given by the SDE

  \[ dU_t = U_t - \frac{dS_t}{S_t} + dP_t. \]

- $S_t = e^{R_t}$ for **Lévy process** $R$ **independent** of **Lévy process** $P$
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- **CTRWs** $R^n$ and $P^n$ (large losses followed by large losses)
  \[ \rightsquigarrow \text{Does the corresponding reserves } U^n \text{ converge?} \]
Application in insurance: ruin theory with investment

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- **CTRWs** $R^n$ and $P^n$ (large losses followed by large losses)
  \( \rightsquigarrow \) Does the corresponding reserves $U^n$ converge?
- Due to the **independence** of $R$ and $P$, one obtains
  \[ U_t = e^{R_t} \left( U_0 + \int_0^t e^{-R_s} dP_s \right) \]
- We can then **show convergence** of
  \[ U^n_t = e^{R^n_t} \left( U^n_0 + \int_0^t e^{-R^n_s} dP^n_s \right) \]