Are linear strategies nearly optimal when trading with superlinear frictions?

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Basic Model

- Single trader with mean variance preferences
- One risky asset and one safe asset with zero interest rate
- Trader receives a noisy signal $p_t$, called the predictor, of future asset returns

$$r_{t+1} = \alpha p_t + \epsilon_{t+1},$$

where

$$p_{t+1} = \rho p_t + \sqrt{1 - \rho^2} \eta_{t+1}$$

follows an AR(1) process with $\rho \in (0, 1)$.

- $\eta$ and $\epsilon$ are iid shocks to the returns and predictor respectively.
- We will denote by $(q_t)_{t \in \mathbb{N}}$ the investors holdings in the risky asset.
- Without costs we are in the setting of classic Markowitz optimization.
Transaction Cost Modelling

- Investor trading affects prices and the investor pays a cost.
- Costs are a function $c : \mathbb{R} \rightarrow [0, \infty)$ of the trade size $\Delta q_t := q_t - q_{t-1}$.
- The investor’s net expected ergodic returns up to a horizon are then given by
  
  $$J(\Delta q) := \lim_{T \to \infty} \mathbb{E} \left[ \frac{1}{T} \sum_{t=0}^{T} (r_{t+1} q_t - c(\Delta q_t)) \right]$$

- We will measure the investors daily dollar risk as the standard deviation of the position size
  
  $$R(\Delta q) := \lim_{T \to \infty} \sqrt{\frac{1}{T} \sum_{t=0}^{T} \sigma^2 \mathbb{E}[q_t^2]},$$

  where $\sigma^2 = \text{Var}(r_t) = \mathbb{E}[\epsilon_t^2]$.
- Mean variance objective: $\max_{\Delta q} (J(\Delta q) - \frac{\lambda}{2} R^2(\Delta q))$ for risk-aversion parameter $\lambda$. 
Common Choices of $c$: Proportionate

$$c(\Delta q) = \gamma_1 |\Delta q|$$

- Induced by bid-offer spreads
- The effects of the costs is invariant with respect to the size of the portfolio and therefore can be studied using small-cost asymptotics even in very general settings
- Effects of the costs is invariant with respect to the size of the portfolio
- Small cost asymptotics tractable and hence serve as relevant approximations (Possamaï, Soner and Touzi, 2013; Martin, 2014; Kallsen and Muhle-Karbe, 2017).
- **Main principle**: optimal to trade as minimally as possible to remain inside “no-trade region” around frictionless optimizer.
Common Choices of $c$: Quadratic Costs

$$c(\Delta q) = \gamma_2 (\Delta q)^2,$$

- **Tractable**: linear quadratic control problem
- Corresponds to linear price impact
- Explicit solutions available (Garleanu & Pedersen 2013)
  - Ergodic case realized as sending the discount factor to zero
  - Explicit solutions available in more general setups (Collin-Dufresne et al., 2020; Ackermann et al., 2022)
- Here the optimal strategy is

$$\Delta q_t^* = K_p \rho_t + (K_q - 1) q_{t-1}^*, \quad t \in \mathbb{N}, \quad (1)$$

where

$$K_p = \frac{\alpha}{2} \frac{\rho}{\gamma_2 (1 - \rho)} + \xi, \quad K_q = \frac{\gamma_2}{\gamma_2 + \xi}, \quad \xi = \frac{\lambda \sigma^2}{4} \left(1 + \sqrt{1 + \frac{8 \gamma_2}{\lambda \sigma^2}}\right)$$
Common Choices of $c$: Power Costs

\[ c(\Delta q) = \gamma_p|\Delta q|^p, \quad p \in (1, 2) \]

- Superlinear and subquadratic regime,
- Leads to concave, rather than linear, price impact,
- A wealth of literature studies the nonlinearity (concavity) of price impact both empirically and theoretically (Mastromatteo et al. (2014a); Gabaix et al. (2003); Tóth et al. (2011); Farmer et al. (2011); Mastromatteo et al. (2014b).)
- The general empirical consensus is that $p \in [1.4, 1.7]$ for a variety of settings. (Almgren et al. (2005); Bershova and Rakhlin (2013); Mastromatteo et al. (2014a); Brokmann et al. (2015); Zarinelli et al. (2015))
  - For our numerical experiments we take $p = 1.6$ as in (Almgren et al. (2005))
Power Law Costs Cont.

• Tractability can be obtained in the small cost asymptotic case (Guasoni and Weber (2020); Cayé et al. (2020)) or in the case of a risk-neutral trader.

• However the general case, where both the risk and the costs are impactful, is **not tractable**.

• This is the case of interest to many funds and active managers of mid-large size.

• **Questions:**
  – Efficient, fast and cost effective ways to solve the problem?
  – Can we obtain approximately optimal strategies? Do they perform well for relevant parameter choices?
  – Is there a “clever” sub-class of strategies one can optimize over?
The Problem Revisited

\[
\max_{\Delta q} \lim_{T \to \infty} \mathbb{E} \left[ \frac{1}{T} \sum_{t=0}^{T} \left( r_{t+1} q_t - \frac{\lambda \sigma^2}{2} q_t^2 - \gamma_p |\Delta q_t|^p \right) \right].
\]

- **Idea:** using our domain knowledge, look for a small parametric class to optimize over to obtain a solution that (hopefully) performs nearly as well as the true optimum.
- Moallemi and Sağlam (2017) in a significantly more general setting look over all linear strategies and show that optimizing over that class is a convex problem.
- In this particular setting we go even further taking a one parameter family of strategies.
LQR Strategies

• Look for the simplest family that achieves high performance
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• Since the cost structure is different we view, $\gamma$ the cost parameter’s contribution to the optimal quadratic strategy as a variable.
• An LQR strategy is one of the form

$$
\Delta q_t(\gamma; \lambda) = K_p(\gamma; \lambda)p_t + (K_q(\gamma; \lambda) - 1)q_{t-1}, \quad t \in \mathbb{N},
$$

where

$$
K_p(\gamma; \lambda) = \frac{\alpha}{2} \frac{\rho}{\gamma(1 - \rho)} + \xi, \quad K_q(\gamma; \lambda) = \frac{\gamma}{\gamma + \xi}, \quad \xi = \frac{\lambda \sigma^2}{4} \left(1 + \sqrt{1 + \frac{8\gamma}{\lambda \sigma^2}}\right)
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LQR Strategies

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- With quadratic costs $q(\gamma_2; \lambda)$ is optimal over all strategies.
Theoretical Performance

- We can compute the performance of any LQR strategy explicitly.

**Theorem (Brokmann, I., Muhle-Karbe, Schmidt, ’23)**

For any $\gamma > 0$, we have

$$J(\Delta q(\gamma; \lambda)) = \frac{\alpha K_p(\gamma; \lambda)}{1 - K_q(\gamma; \lambda)\rho} - \gamma p \sigma^p(\gamma; \lambda) \frac{2^{p/2} \Gamma \left(\frac{p+1}{2}\right)}{\sqrt{\pi}},$$

where

$$\sigma^2(\gamma; \lambda) = K_p^2(\gamma; \lambda) \left(1 - \frac{2\rho(1 - K_q(\gamma; \lambda))}{1 - K_q(\gamma; \lambda)\rho} + \frac{(1 - K_q(\gamma; \lambda))^2(1 + K_q(\gamma; \lambda)\rho)}{(1 - K_q^2(\gamma; \lambda))(1 - K_q(\gamma; \lambda)\rho)}\right).$$

Moreover, the corresponding daily dollar risk is

$$R(\Delta q(\gamma; \lambda)) = \sqrt{\frac{\sigma^2 K_p^2(\gamma; \lambda)(1 + K_q(\gamma; \lambda)\rho)}{(1 - K_q^2(\gamma; \lambda)\rho)(1 - K_q(\gamma; \lambda)\rho)}}.$$
Benchmark

• We will restrict to LQR strategies, but would like to have a benchmark for comparison.

• We use the **Viterbi algorithm** first proposed by Viterbi (1967) and used by Kolm & Ritter (2014) in a transaction cost setting to solve for the true optimum.

• This algorithm can be adapted to any cost function, multiple assets (but with single-asset constraints), time dependent coefficients etc.

• We first test the algorithm on the problem with quadratic costs to ascertain its implementation.

• Then we run it until it converges up to a small tolerance level for the $p = 1.6$ cost case.
Risk Matching

• Many practitioners have a daily dollar risk (DDR) target for their strategies.
• To compare apples to apples we first obtain the DDR for the optimal strategy estimated by the Viterbi algorithm.
• Then for any $\gamma > 0$ we choose the risk-aversion parameter $\lambda(\gamma)$ so that the strategy $q(\gamma) := q(\gamma; \lambda(\gamma))$ has the required risk.
• Not explicitly solvable, but easily done numerically as it is a scalar equation.
MATS Estimator

• Simplest choice for $\gamma$: match the “effective” quadratic cost to the true subquadratic cost.
• That is take

$$\gamma = \gamma_p \frac{\mathbb{E}[|\Delta q^*|]^p}{\mathbb{E}[|\Delta q^*|]^2},$$

where $\mathbb{E}[|\Delta q^*|]$ is calculated on the optimal trades obtained from the optimal quadratic cost position.
MATS Estimator

Impact costs for non-linear impact and the MATS estimation

- Linear impact
- Non-linear impact

Trade size (kUSD)
Trading costs (a.u.)
$\varepsilon[|u_t|]$

0 100 300 400
0.0 0.5 1.0 1.5 2.0 2.5 3.0 3.5 4.0
MATS Estimator

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where $\mathbb{E}[|\Delta q^*|]$ is calculated on the optimal trades obtained from the optimal quadratic cost position.

• Does not work very well!

• Roughly 10%-30% lower Sharpe for relevant parameters and daily dollar risks.
One parameter LQR Family

- Next simplest choice: optimize over $\gamma$.
- Given a DDR level we find for every $\gamma > 0$, $\lambda(\gamma)$ such that $R(\Delta q(\gamma, \lambda(\gamma))) = \text{DDR}$.
- That we solve
  $$\max_{\gamma>0} J(\Delta q(\gamma; \lambda(\gamma))).$$
- Since the risk is fixed this is equivalent to maximizing the Sharpe ratio
  $$\max_{\gamma>0} \text{SR}(\gamma) = \max_{\gamma>0} \frac{J(\Delta q(\gamma, \lambda(\gamma)))}{\text{DDR}}.$$ 
- Due to the theorem this is a scalar maximization problem!
- Explicit solution not available, but is numerically simple and very fast to implement.

*How well does it do?*
Numerical Results

LQR Performance

Daily Dollar Risk: 500K

(0.74, 1.92)  1.93

Daily Dollar Risk: 1.3M

(0.57, 1.56)  1.58

Daily Dollar Risk: 4.8M

(0.39, 0.97)  0.99
Observations

Parameters:

\[ p = 1.6, \quad \sigma = 0.02, \quad \alpha = 1.67 \times 10^{-4} \]
\[ \gamma_p = 1.38 \times 10^{-7} = 0.079 \frac{\sigma}{\text{ADV}^{0.6}} \quad \text{ADV} = 10\text{M}. \]

- **LQR performs nearly as well as the optimum!**
  - Loss in Sharpe from LQR restriction between 0.5%-2%
  - Growing in DDR?
- **Optimal LQR parameter is substantially below the MADS estimator**
  - Jensen/concavity effect.
  - Decreasing with daily dollar risk.
- **The performance is quite sensitive to parameter optimization especially for higher risk levels.**
Conclusion and Future Work

• Even with nonlinear impact, linear strategies can be nearly optimal.
• **Domain knowledge** helps in finding a suitable parametric class that is a one-parameter subset of the class of linear strategies.
• The **effective cost** is substantially lower than the matched cost at the mean.
• Method extendable to **multi-asset setting**.
• Performance guarantees in terms of parameters?
• Incorporating impact decay as in Obizhaeva & Wang (2013)?
Stochastic Control’s Sonnet Symphony

In Princeton’s halls, where brilliance is found,
We gather, minds aflame, in knowledge’s quest.
Mean-variance preferences, truth unbound,
Noisy signals, transaction costs put to test.

Linear strategies, their optimality renowned,
Quadratic frictions, Garleanu and Pedersen’s best.
Yet, superlinear costs, a challenge unbound,
No explicit solutions, minds put to the test.

But fear not, for a class of linear grace,
One parameter, simplicity embraced.
Nearly optimal performance, a shining trace,
Brute force, our guide, a strategy interlaced.

As we bid farewell to this learned session,
Stochastic control, a journey in progression.

Thank you!
Viterbi Algorithm

- Fix a sample path of the predictor.
- Sample sequences of positions \((q_{i}^{'})_{i=1,...,\kappa}\) and use Bellman’s condition

\[
\nu_{t+1}(q_{t+1}^{'}) = \max_{q_{t}}[\log p(q_{t+1}^{*}|q_{t}^{'}) + \log p(q_{t+1}^{'}|q_{t}^{'}) + \nu_{t}(q_{t}^{'})]
\]  \hspace{1cm} (2)

where

\[
p(q^{*}, q^{'}) = \frac{\lambda \sigma^{2}}{2} ((q_{t}^{*})^{2} + (q_{t}^{'})^{2}) - \alpha q_{t}^{'} \rho_{t}
\]

to develop a criterion for how “good” the sample path is.
- Forward populate large array using the right hand side of (2).
- Find optimal path by backpropogating.