Moral hazard for time-inconsistent agents, BSVIEs and stochastic targets

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Outline

1. The big picture
   - Motivating example
   - Time-inconsistency

2. Main results
   - An extended DPP
   - The characterising BSDE system

3. Back to contract theory
The setting

- Setting similar to Hölstrom and Milgrom (1987). Agent chooses control $\alpha \in \mathcal{A}$, and induces probability measure $\mathbb{P}^\alpha$ such that

$$X_t = x + \int_0^t \alpha_s ds + \sigma W_t^\alpha, \ t \in [0, T].$$
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• Principal rewards Agent at time $T$ with $\xi(X_{\wedge T})$. Agent’s criterion is

$$J(t, \alpha, \xi) := \mathbb{E}^{\mathbb{P}^{\alpha}} \left[ f(T - t)\xi(X_{\wedge T}) - \int_t^T f(s - t)\frac{\alpha_s^2}{2} ds \bigg| \mathcal{F}_t \right],$$

with $f(0) = 1$. 

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with $f(0) = 1$.

- Given effort $\alpha$ and contract $\xi$, Principal's criterion is

$$\mathbb{E}^{P^\alpha} [X_T - \xi(X_{\wedge T})].$$
Exponential discounting case

- Standard choice: \( f(x) := e^{-\delta x} \), for some \( \delta \geq 0 \). In this case, dynamic programming principle holds. Agent can be taken as a utility maximiser.
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- Developed with Cvitanić and Touzi a general approach to find optimal contracts. Applied here, implies that there is some process \( Z \) such that

\[
Y_t = \xi(X_{\wedge T}) + \int_t^T \left( \frac{Z_s^2}{2} - \delta Y_s \right) ds - \int_t^T Z_s dX_s,
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where \( Y \) is the value function of Agent.
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- **Stochastic representation** for the value function of Agent, and thus for \( \xi(X_{t,T}) \).
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where \( Y \) is the value function of Agent.

- Stochastic representation for the value function of Agent, and thus for \( \xi(X_{\cdot \wedge T}) \).

- Principal then maximises his criterion over \( Z \), optimal contract is

\[
\xi(X_{\cdot \wedge T}) = C + X_T,
\]

with \( C \) chosen to ensure that Agent accepts the contract.
Beyond exponential discounting?

- All previous results rely on DPP, and thus require exponential discounting...
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  (i) First task: understand problem faced by Agent \( \implies \) study generic time-inconsistent stochastic control problems
Beyond exponential discounting?

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● What should we do?

   (i) First task: understand problem faced by Agent $\implies$ study generic time-inconsistent stochastic control problems

   (ii) Second task: understand problem faced by Principal $\implies$ doable in the example, general case still open, but seems linked to optimal control of stochastic Volterra equations.
The basic problem

- On space \((\Omega, \mathcal{F})\), let \(P^\nu\) be a weak solution to the controlled SDE

\[
X_t = x + \int_0^t b_r(X_r \wedge \cdot, \nu_r)dr + \int_0^t \sigma_r(X_r \wedge \cdot) dW_r, \quad t \in [0, T].
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The basic problem

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$$X_t = x + \int_0^t b_r(X_{r\wedge.}, \nu_r)dr + \int_0^t \sigma_r(X_{r\wedge.})dW_r, \ t \in [0, T].$$

• The reward functional

$$\nu(t, \nu) := J(t, t, \nu) = \mathbb{E}^{\mathbb{P}^\nu}\left[ \int_t^T f_r(t, X_{r\wedge.}, \nu_r)dr + F(t, X_{T\wedge.}) \mid \mathcal{F}_t \right].$$
• On space \((\Omega, \mathcal{F})\), let \(\mathbb{P}^\nu\) be a \textit{weak solution} to the controlled SDE

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\nu(t, \nu) := J(t, t, \nu) = \mathbb{E}^{\mathbb{P}^\nu}\left[ \int_t^T f_r(t, X_{r\wedge \cdot}, \nu_r)dr + F(t, X_{T\wedge \cdot}) \bigg| \mathcal{F}_t \right].
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• \textbf{Control problem:} Since DPP fails, look for equilibria.
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Though classical DPP does not hold, can obtain

**Theorem [Hernández, P. (2019)]**

If $\nu^*$ is an equilibrium then

$$
\nu(0, \nu^*) = \sup_{\nu \in \nu} \mathbb{E}^{P^\nu} \left[ \nu(\tau, \nu^\tau) + \int_0^\tau \left( f_r(r, X_r\wedge\cdot, \nu_r) - \frac{\partial F}{\partial s}(r) - \int_r^T \frac{\partial f_r}{\partial s}(u, X_r\wedge\cdot, \nu_u^*) du \right) dr \right].
$$
Define the Hamiltonian

\[ H_t(x, z) := \sup_{a \in A} \{ h_t(t, x, z) \}, \quad h_t(s, x, z, a) := f_t(s, x, a) + b_t(x, a) \cdot z, \]

\[ \nu_t^*(x, z) := \arg\max_{a \in A} \{ h_t(t, x, z, a) \}. \]
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Extended DPP relates value of the agent at equilibrium to
\[
\begin{cases}
Y_t = F(T, X_{\wedge T}) + \int_t^T (H_r(X, Z_r) - \partial Y^s_r) dr - \int_t^T Z_r \cdot dX_r, \\
Y^s_t = F(s, X_{\wedge T}) + \int_t^T h_r(s, X, \nu^*_r(X_{\wedge r}, Z_r), Z^s_r) dr - \int_t^T Z^s_r \cdot dX_r,
\end{cases}
\]
where
\[ \partial Y^s_t := \frac{\partial}{\partial s} Y^s_t. \]
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Extended DPP relates value of the agent at equilibrium to

\[
\begin{align*}
Y_t &= F(T, X_{\cdot \wedge T}) + \int_t^T (H_r(X, Z_r) - \partial Y_r^s)dr - \int_t^T Z_r \cdot dX_r, \\
Y_t^s &= F(s, X_{\cdot \wedge T}) + \int_t^T h_r(s, X, \nu_r^*(X_{\cdot \wedge r}, Z_r), Z_r^s)dr - \int_t^T Z_r^s \cdot dX_r,
\end{align*}
\]

where

\[ \partial Y_t^s := \frac{\partial}{\partial s} Y_t^s. \]

For exponential discounting, \( \partial Y_t^s = \delta Y_s \implies \text{no coupling} \) and classical HJB BSDE.
HJB BSDE system (2)

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- **Non-Markovian version** of the **non-local PDE system** of Ekeland and Lazrak (2008).
- Earlier contributions argued by **passing informally to the limit from discrete-time.** Thanks to extended DPP

**Theorem [Hernández, P. (2019)]**

*If $\nu^*$ is an equilibrium, then there exists a solution to the BSDE system, and necessarily $\nu^* = \nu^*(X, Z)$.***
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Agent's problem

- **Agent** looks for equilibria, so that **Principal** can only offer \( \xi(X \land T) \) such that at least one exists. By our results, equilibria exist if and only if we have a solution to

\[
Y_t = \xi(X \land T) + \int_t^T \left( \frac{Z_r^2}{2} - \partial Y'_r \right) dr - \int_t^T Z_r dX_r,
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\[
Y_t^s = f(T - s)\xi(X \land T) + \int_t^T \left( Z_r Z^s_r - f(r - s) \frac{Z_r^2}{2} \right) dr - \int_t^T Z^s_r dX_r,
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the equilibrium is \( \nu^* := Z \), and \( J(t, \alpha^*, \xi) = Y_t = Y^t_t \).
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the equilibrium is $\nu^* := Z$, and $J(t, \alpha^*, \xi) = Y_t = Y_t^t$.

- Infinitely many representations for $\xi(X,^\wedge T)$...

- ...cannot use only one and optimise over $Z$ as before, need to understand relationships between $Z$ and $Z^s$. 
We can directly check that

\[ Z_t = Z^t, \text{ and } Z^s_t = \frac{f(T - s)}{f(T)} Z^0_t + f(T - s) \tilde{Z}^s_t, \]

where \( \tilde{Z}^s \) appears in the following martingale representation

\[ M^s_t := \mathbb{E}_{\mathbb{P}^\nu^*} \left[ \int_0^T \left( \frac{f(r)}{f(T)} - \frac{f(r - s)}{f(T - s)} \right) \frac{Z^2_r}{2} dr \bigg| \mathcal{F}_t \right] = M^s_0 + \int_0^t \tilde{Z}^s_r dX_r. \]
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• Notice that when \( f(t) := e^{-\delta t} \), the second term vanishes \( \rightarrow \) this is the effect due to time-inconsistency.
The BSDE system

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- It also vanishes whenever \( Z \) is deterministic!
Solution

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$$
\xi^* := C + \int_0^T \frac{f(T)}{f(r)f(T-r)} \mathrm{d}X_r.
$$

- Can check directly that this is an admissible contract which attains an upper bound for Principal's value.
The general case?

In great generality, system is equivalent to solving, \((s, t) \in [0, T]^2\)

\[
Y_t^s = U_A(s, \xi) + \int_t^T h_r^*(s, X_r^{\wedge \cdot}, Y_r^s, Z_r^s, Y_r^r, Z_r^r) dr - \int_t^T Z_r^s dX_r.
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\]

\(\mathcal{H}^{2,2} : Z \in \overline{H}^{2,2}\) satisfying

\[
Y_t^{s,Z} = Y_0^s - \int_0^t h_r^*(s, X_r \wedge, Y_r^{s,Z}, Z_r^s, Y_r^{r,Z}, Z_r^r) \, dr + \int_0^t Z_r^s \, dX_r, \ (s, t) \in [0, T]^2.
\]

\[
U_A^{(-1)}(s, Y_T^s) = U_A^{(-1)}(u, Y_T^u), \ (s, u) \in [0, T]^2.
\] (3)
To sum up

- Insight: $V^A$ solves a **backward stochastic Volterra integral equation**.

- Principal’s problem boils to optimal control of a **Volterra forward equation (1)** with **Volterra controls (2)**, and **stochastic target constraints (3)** i.e. a family $(Z^s_t)_{(s,t) \in [0,T]^2}$, (3) holds

$$Y^s_t = U_A(s, \xi) + \int_t^T h^*_r(s, X_r \cdot, Y^s_r, Z^s_r, Y^r_r, Z^r_r)dr - \int_t^T Z^s_r dX_r.$$  

- (1) is known, (2) is expected to be a slight generalisation, (3) is the really hard part: need to understand Volterra stochastic target problems, and optimal control of Volterra processes with state constraints.
Stochastic target constraints seem to be pervasive in these problems! In their standard (non-Volterra) form, they appear when

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**Stochastic target constraints** seem to be pervasive in these problems! In their standard (non-Volterra) form, they appear when

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- **Agent** has mean–variance type criteria;
- more surprisingly: when **Agent** and **Principal** play a **Nash game** between themselves;
Stochastic target constraints seem to be pervasive in these problems! In their standard (non-Volterra) form, they appear when

- constraints are put on payments $\xi$;

- Agent has mean–variance type criteria;

- more surprisingly: when Agent and Principal play a Nash game between themselves;

- conjecture this holds true generically for Stackelberg games in continuous-time: problem of the leader boils down to optimal control with stochastic target constraints.
Thank you for your attention!