Multi-agent Targeted Trading Equilibrium with Transaction Costs

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Stylized two-agent models: W. (2018), Noh and W. (2021), Loewenstein and Qin (2022)

Approximate equilibria with two agents: Herdegen and Muhle-Karbe (2018), Gonon, Muhle-Karbe, Shi (2021)


Quadratic penalties: Bouchard et. al. (2018), Herdegen, Muhle-Karbe, Possamaï (2021)

Goal: to prove the existence of a financial equilibrium with proportional transaction costs for an arbitrary, finite number of agents
Model Set-up

Consider a model with a single consumption good (the numeraire) and times \( t \in [0,T]. \) \( B \) is a Brownian motion.

**Two Assets:** a **bank account** with zero interest rate and a **stock** that pays a dividend \( D \) at time \( T. \) We seek the price process \( S = (S_t)_{t\in[0,T]} \) in equilibrium of the form:

\[
dS_t = \mu_t \, dt + \sigma_t \, dB_t, \quad S_T = D.
\]

The volatility \( \sigma \) comes directly from the Itô representation:

\[
D = \mathbb{E}D + \int_0^T \sigma_t \, dB_t,
\]

and \( \mu \) is to be determined in equilibrium.
I ≥ 3 Investors

Trading strategies: \( \theta = (\theta_t)_{t \in [0,T]} \) are adapted, càdlàg, of finite variation on \([0,T]\),
\[
\theta_t = \theta_t^\uparrow - \theta_t^\downarrow, \quad t \in [0,T].
\]

Transaction costs: The investors pay transaction costs that are proportional to the number of shares traded. For strategy \( \theta \), the associated wealth process is
\[
X^\theta_t = \int_0^t \theta_u \, dS_u - \lambda \left( \theta_t^\uparrow + \theta_t^\downarrow \right).
\]

Penalties: Investor \( i \) has trading target \( a_i \) (constant) and inventory penalties \( L^\theta_i \) paid at time \( T \),
\[
L^\theta_{i,T} = \frac{1}{2} \int_0^T \kappa(u) (\gamma(u)a_i - \theta_u)^2 \, du,
\]
where \( \kappa : [0,T] \rightarrow (0,\infty) \) is the intensity of the penalty, \( \gamma : [0,T] \rightarrow [0,1] \) is the desired trading target trajectory. \( \gamma \) is continuous, strictly increasing, \( \gamma(0) = 0, \gamma(T) = 1 \).
**Equilibrium Definition**

**Equilibrium** consists of a continuous Itô process $S = (S_t)_{t \in [0,T]}$ and trading strategies $\theta_1, \ldots, \theta_I$ such that

- **Optimality:** For $1 \leq i \leq I$, the strategy $\theta_i$ is optimal for
  \[
  \mathbb{E} \left[ X_T^{\theta_i} - L_{i,T}^{\theta_i} \right] = \sup_{\theta} \mathbb{E} \left[ X_T^{\theta} - L_{i,T}^{\theta} \right],
  \]
  where
  \[
  X_T^{\theta} = \int_0^T \theta_u dS_u - \lambda \left( \theta_T^\uparrow + \theta_T^\downarrow \right), \quad L_{i,T}^{\theta} = \frac{1}{2} \int_0^T \kappa(u) \left( \gamma(u) a_i - \theta_u \right)^2 du.
  \]

- **Market Clearing:** We have $\sum_{i=1}^I \theta_{i,t} = 0$ for all $t \in [0,T]$.
- **Terminal Stock Price Constraint:** $S_T = D$. 

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Transaction Costs in Equilibrium
We conjecture that the equilibrium stock price drift is *deterministic* $\mu = \mu(t)$. Given a deterministic stock price drift $\mu$, for each investor $i$, we seek a strategy $\theta$ such that $(\theta, Y_i^\theta)$ satisfies

$$Y_{i,t}^\theta = \int_t^T \kappa(u) \left( \frac{\mu(u)}{\kappa(u)} + \gamma(u)a_i - \theta_u \right) du, \quad t \in [0,T],$$

$$-\lambda \leq Y_{i,t}^\theta \leq \lambda, \quad t \in [0,T],$$

$$0 = \int_0^T (\lambda - Y_{i,t}^\theta) \, d\theta_t^\uparrow = \int_0^T (\lambda + Y_{i,t}^\theta) \, d\theta_t^\downarrow.$$

**Difficulties:**

- Optimal strategies $\theta_1, \ldots, \theta_I$ must clear: $\sum_i \theta_{i,t} = 0$ for $t \in [0,T]$.
- $\mu$ is determined endogenously via market clearing.

In Noh and W. (2021), $I = 2$ and $u \mapsto \frac{\mu(u)}{\kappa(u)} + \gamma(u)a_i$ is monotone.
Closer Look at the First-Order Condition (II)

Given a stock price drift $\mu = \mu(t)$, for each investor $i$, we seek a strategy $\theta$ such that $(\theta, Y^\theta_i)$ satisfies

$$Y^\theta_{i,t} = \int_t^T \kappa(u) \left( \frac{\mu(u)}{\kappa(u)} + \gamma(u) a_i - \theta_u \right) du, \quad t \in [0,T],$$

$$-\lambda \leq Y^\theta_{i,t} \leq \lambda, \quad t \in [0,T],$$

$$0 = \int_0^T (\lambda - Y^\theta_{i,t}) d\theta^\uparrow_t = \int_0^T (\lambda + Y^\theta_{i,t}) d\theta^\downarrow_t.$$

If $u \mapsto \frac{\mu(u)}{\kappa(u)} + \gamma(u) a_i$ is monotone and has the same sign for all $u \in [0,T]$, then:

- $t \mapsto \int_t^T \kappa(u) \left( \frac{\mu(u)}{\kappa(u)} + \gamma(u) a_i \right) du$ attains its largest magnitude at $t = 0$.

- Expect to have $\theta_{i,t} = \frac{\mu(t)}{\kappa(t)} + \gamma(t) a_i$ when trade occurs and note that trade is monotone.

- Buying/selling constraints tell us when trade should stop.
Conjectured Form of Equilibrium

For each investor \( i \), we conjecture that there exists a *stop-trade time* \( \tau_i \in [0, T) \) where investor \( i \)'s optimal trading strategy is

\[
\theta_{i,t} = \begin{cases} 
\frac{\mu(t)}{\kappa(t)} + \gamma(t)a_i, & t \in [0, \tau_i], \\
\theta_{i,\tau_i}, & t \in (\tau_i, T].
\end{cases}
\]

Then, \( Y_{t}^{\theta_i} \) is constant for \( t \in [0, \tau_i] \).

Suppose that the trading targets are distributed well enough so that

\[
0 =: \tau_0 \leq \tau_1 \leq \ldots \leq \tau_{I-2} \leq \tau_{I-1} = \tau_I < T.
\]

Then the market clearing condition for \( t \in (\tau_i, \tau_{i+1}] \) is

\[
0 = \sum_{k=1}^{i} \theta_{k,t} + \sum_{k=i+1}^{I} \theta_{i,t} = \sum_{k=1}^{i} \theta_{k,\tau_k} + \sum_{k=i+1}^{I} \left( \frac{\mu(t)}{\kappa(t)} + \gamma(t)a_k \right)
\]

\[
\implies \mu(t) = -\frac{\kappa(t)}{I-i} \left( \gamma(t) \sum_{k=i+1}^{I} a_k + \sum_{k=1}^{i} \theta_{k,\tau_k} \right), \quad t \in (\tau_i, \tau_{i+1}].
\]
Equilibrium Trading Strategies

Investor $i$ monotonically trades on $[0, \tau_i]$ and does not trade on $(\tau_i, T]$.

Figure: $I = 20$, $T = 1$, $\lambda = 0.2$, $\kappa(t) = 0.1$, $\gamma(t) = t/T$.
Equilibrium Stock Price Drift

Figure: $I = 20$, $T = 1$, $\lambda = 0.2$, $\kappa(t) = 0.1$, $\gamma(t) = t/T$
Equilibrium Trading Intervals

Figure: $I = 20$, $T = 1$, $\lambda = 0.2$, $\kappa(t) = 0.1$, $\gamma(t) = t/T$
Rank-Based Ordering (I)

**Goal:** Rearrange the trading targets \((a_1, \ldots, a_I)\) to a *rank-based ordering of targets* \((a^{(1)}, \ldots, a^{(I)})\), where \(a^{(j)}\) refers to the \(j^{th}\) trader to stop trading so that the trading times are ordered by

\[
0 \leq \tau^{(1)} \leq \tau^{(2)} \leq \ldots \leq \tau^{(I-1)} = \tau^{(I)} < T.
\]

Construct the rank-based order by *backward induction:*

**Base Case:** Let \(\mathcal{I}^{(I-1)} := \{1, \ldots, I\}\). For \(i,l \in \mathcal{I}^{(I-1)}\), we put

\[
\eta^{(I-1)}_{i,l} := \inf \left\{ t \in [0,T] : \left( a_i - \frac{1}{2} (a_l + a_i) \right) \int_t^T \kappa(u) \left( \gamma(u) - \gamma(t) \right) du \leq \lambda \right\}.
\]

We set

\[
(i^*, l^*) \in \text{Argmax} \left\{ \eta^{(I-1)}_{i,l} : i,l \in \mathcal{I}^{(I-1)} \right\},
\]

\[
 a^{(I-1)} := a_i^*, \quad a^{(I)} := a_{l^*}, \quad A^{(I-1)} := a^{(I-1)} - \frac{1}{2} (a^{(I-1)} + a^{(I)}) =: -A^{(I)},
\]

\[
 \tau^{(I)} := \tau^{(I-1)} := \eta^{(I-1)}_{i^*, l^*}, \quad \mathcal{I}^{(I-2)} := \mathcal{I}^{(I-1)} \setminus \{i^*, l^*\}.
\]
Backward induction: Let \( j \) be given with \( 1 \leq j \leq I - 2 \). We assume that the following have been previously defined: \( \mathcal{I}(j), \ldots, \mathcal{I}(I-1); \tau(j+1), \ldots, \tau(I); a^{(j+1)}, \ldots, a^{(I)}; \) and \( A^{(j+1)}, \ldots, A^{(I-1)} \). For \( i \in \mathcal{I}(j) \), we define

\[
\eta_i^{(j)} := \inf \left\{ t \in [0,T] : \left| \left( a_i - \frac{1}{I-j-1} \sum_{k=j+1}^{I} a_k \right) F(t) + \sum_{k=j+1}^{I-2} \frac{A_k}{I-k} F(t \vee \tau^{(k)}) \right| \leq \lambda \right\}.
\]

We set

\[
i^* \in \text{Argmax} \left\{ \eta_i^{(j)} : i \in \mathcal{I}(j) \right\}, \quad a^{(j)} := a_{i^*}, \quad \tau^{(j)} := \eta_{i^*}^{(j)},
\]

\[
A^{(j)} := a^{(j)} - \frac{1}{I-j+1} \sum_{k=j}^{I} a^{(k)}, \quad \mathcal{I}(j-1) := \mathcal{I}(j) \setminus \{i^*\}.
\]
Construction of Equilibrium

Candidate stock price drift:

\[
\mu(t) := \begin{cases} 
-\kappa(t) \left( \frac{\gamma(t)}{I-j} \sum_{k=j+1}^{I} a^{(k)} + \sum_{k=1}^{j} \frac{\gamma(\tau^{(k)}) A^{(k)}}{I-k} \right), & t \in [\tau(j), \tau(j+1)), \\
-\kappa(t) \left( \frac{\gamma(t)}{2} \sum_{k=I-1}^{I} a^{(k)} + \sum_{k=1}^{I-2} \frac{\gamma(\tau^{(k)}) A^{(k)}}{I-k} \right), & t \in [\tau(I-1), T].
\end{cases}
\]

Candidate optimal trading strategy:

\[
\theta_{t}^{(j)} := \begin{cases} 
\frac{\mu(t)}{\kappa(t)} + \gamma(t) a^{(j)}, & t \in [0, \tau(j)), \\
\theta_{\tau(j)}^{(j)}, & t \in (\tau(j), T].
\end{cases}
\]

Candidate first-order condition processes:

\[
Y_{t}^{(j)} := \int_{t}^{T} \kappa(u) \left( \frac{\mu(u)}{\kappa(u)} + \gamma(u) a^{(j)} - \theta_{u}^{(j)} \right) du, \quad t \in [0, T].
\]
Theorem (Choi, Duraj, W. (2022))

There exists an equilibrium in which the stock price is given by

\[ S_t := \mathbb{E}[D] + \int_0^t \sigma_u dB_u - \int_t^T \mu(u) du, \quad t \in [0,T], \]  

and \( \theta^{(1)}, \ldots, \theta^{(I)} \) are the optimal trading strategies.

Ingredients of proof: We prove the following properties:

- \( \tau^{(1)} \leq \tau^{(2)} \leq \cdots \leq \tau^{(I)} \)
- \( \left| Y_t^{(j)} \right| \leq \lambda \)
- If \( \tau^{(j)} > 0 \) and \( A^{(j)} \geq 0 \), then
  - \( Y_t^{(j)} = \lambda \) for \( t \in [0,\tau^{(j)}] \)
  - \( \theta_t^{(j)} = \theta_T^{(j)} \) for \( t \in [\tau^{(j)},T] \)
  - \( \theta_t^{(j)} \downarrow = 0 \) for \( t \in [0,T] \)
  - \( \sum_{j=1}^I \theta_t^{(j)} = 0 \) for \( t \in [0,T] \)
Properties of the equilibrium

- The rearrangement \((a^{(j)})_{1 \leq j \leq I}\) only depends on \((a_i)_{1 \leq i \leq I}\), not on \(\gamma, \kappa, \) or \(\lambda\).

\[ \frac{d}{dt} \left[ \frac{\mu(t)}{\kappa(t)} \right] = \begin{cases} 
-\gamma'(t) \sum_{k=j+1}^{I} a^{(k)}, & t \in (\tau(j), \tau(j+1)), 0 \leq j \leq I - 2, \\
-\frac{\gamma'(t)}{2} \sum_{k=I-1}^{I} a^{(k)}, & t \in (\tau(I-1), T]. 
\end{cases} \]
Properties of the equilibrium

- The map \( \lambda \mapsto S_0(\lambda) \) is piecewise linear.
Private Trading Targets (I)

What if the investors’s trading targets $a_1, \ldots, a_I$ are private?

Investor $i$ knows $F_{i,t} = \sigma (\mu_s, \sigma_s, a_i : 0 \leq s \leq t)$ at time $t$ and optimizes

$$
\sup_{\theta} \mathbb{E} \left[ X_T^\theta - \frac{1}{2} \int_0^T \kappa(u) (\gamma(u)a_i - \theta_u)^2 \, du \mid F_{i,t} \right].
$$

**First-order condition:** seek a strategy $\theta$ such that $(\theta, Y_{i,t}^\theta)$ satisfies

$$
Y_{i,t}^\theta = \mathbb{E} \left[ \int_t^T \kappa(u) \left( \frac{\mu_u}{\kappa(u)} + \gamma(u)a_i - \theta_u \right) \, du \mid F_{i,t} \right], \quad t \in [0,T],
$$

$$
-\lambda \leq Y_{i,t}^\theta \leq \lambda, \quad t \in [0,T],
$$

$$
0 = \int_0^T (\lambda - Y_{i,t}^\theta) \, d\theta_t^\uparrow = \int_0^T (\lambda + Y_{i,t}^\theta) \, d\theta_t^\downarrow.
$$

*buying constraint*  \hspace{1cm} *selling constraint*
Private Trading Targets (II)

Investor $i$ seeks a strategy $\theta$ such that $(\theta, Y_i^{\theta})$ satisfies

$$Y_{i,t}^{\theta} = \mathbb{E} \left[ \int_t^T \kappa(u) \left( \frac{\mu_u}{\kappa(u)} + \gamma(u) a_i - \theta_u \right) du \mid \mathcal{F}_{i,t} \right], \quad t \in [0,T],$$

$$-\lambda \leq Y_{i,t}^{\theta} \leq \lambda, \quad t \in [0,T],$$

$$0 = \int_0^T (\lambda - Y_{i,t}^{\theta}) \, d\theta_t^\uparrow = \int_0^T (\lambda + Y_{i,t}^{\theta}) \, d\theta_t^\downarrow.$$

**Difficulties:**

- $\mu$ is stochastic, and $(\theta, Y_i^{\theta})$ is no longer a deterministic forward-backward system.
- The filtrations $(\mathcal{F}_{i,t})_t$ are *outputs* of equilibrium, not *inputs*.
- The stop-trade times and rank-based ordering (if they exist!) are no longer *deterministic*.
- Still need market clearing: $\sum_i \theta_{i,t} = 0$ for all $t \in [0,T]$. 
THANK YOU!