General Equilibrium with Unhedgeable Fundamentals and Heterogeneous Agents

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Motivation

- How do unhedgeable risk factors impact consumption decisions and asset pricing?
- Important sources of risk cannot be perfectly hedged.
- Most literature on general equilibria assumes dynamically complete markets.
- Market completeness guarantees the existence of a representative agent.
- Models in which the drivers of assets’ fundamentals are unhedgeable have so far eluded academic research.
What we do

- First closed-form expressions for the general equilibrium of a continuous-time market with unhedgeable risk factors and heterogeneous agents.

- Explicit expressions allow us to pin down the mechanism through which unhedgeable shock alter consumption decisions and trading strategies, and how asset prices react in equilibrium.

- **Methodology**: Small-noise asymptotics

- Unhedgeable risk factors have small volatility. Expansion around a complete model admitting a closed-form solution.
Market

- The dividend (or aggregate consumption) process $D_t$ has dynamics
  \[
  dD_t = (\bar{\mu} + \varepsilon \sigma z_t)dt + \sigma_D dW^D_t, \\
  dz_t = -az_t dt + dW^\mu_t.
  \]

- $W^\mu$, $W^D$ independent Brownian motions.

- $z_t$ fundamental variable. Parameter $\varepsilon$ controls fluctuations’ size.
  For simplicity, $z_t$ is a scalar. Extension with multiple factors is possible.

- $\varepsilon = 0$: baseline complete market.

- $\varepsilon > 0$: incomplete market. Fluctuations unhedgeable: one stock, two shocks.
Preferences

- $n$ agents maximize lifetime utility from consumption.
- The $i$-th agent has absolute risk-aversion $\alpha_i$ and a time-preference rate $\beta_i$. Maximizes

$$E \left[ \int_0^\infty e^{-\beta_i s} U_i(c_s) ds \right]$$

- Constant absolute risk aversion: $U_i(c) = \frac{e^{-\alpha_i c}}{-\alpha_i}$. CARA utilities needed to obtain explicit expressions.
- Agents are endowed with some initial wealth $x_i$.
- Agents finance consumption by:
  - Borrowing from and lending to each other
  - Earning dividends and trading the dividend-paying assets.
Budget Equation

- $i$-th agent’s holds $\theta^i_t$ shares of stock at time $t$. Consumes at rate $c^i_t$.
- Budget equation:

$$dX^i_t = \theta^i_t(dP_t + D_t\,dt) + r_t(X_t - \theta^i_t P_t)\,dt - c^i_t\,dt \quad X^i_0 = x_i$$

- Wealth $X^i_t$ of $i$-th agent changes from
  - capital gains and losses $\theta^i_t \,dP_t$;
  - dividends $\theta^i_t D_t\,dt$;
  - interest income and charges $r_t(X_t - \theta^i_t P_t)\,dt$;
  - consumption $c^i_t\,dt$.

- Stock price $P_t$ and interest rate $r_t$ endogenous.
Second-Order Equilibrium

A second-order equilibrium is a family \( \{r_t(\varepsilon), P_t(\varepsilon), (\hat{c}^i_t(\varepsilon))_{1 \leq i \leq n}, (\hat{\theta}^i_t(\varepsilon))_{1 \leq i \leq n}\}_{0 \leq \varepsilon \leq \bar{\varepsilon}} \) such that

1. for every \( \varepsilon \), the market clearing conditions hold:
   \[
   \sum_{i=1}^{n} \theta^i_t = 1, \quad \sum_{i=1}^{n} (X^i_t - \theta^i_t P_t) = 0, \quad \sum_{i=1}^{n} \hat{c}^i_t = D_t \quad \text{a.s. for all } t > 0
   \]

2. for every \( \varepsilon \), agent \( i \)'s consumption is financed by their trading strategy.

3. for every agent \( i \), consumption is second-order optimal in \( \varepsilon \), i.e.,
   \[
   E \left[ \int_0^\infty e^{-\beta_i t} U_i(\hat{c}^i_t) dt \right] \geq \sup_{(c_t, \theta_t) \in A} E \left[ \int_0^\infty e^{-\beta_i t} U_i(c_t) dt \right] + o(\varepsilon^2).
   \]
Baseline Model

- When $\varepsilon = 0$, dividend growth rate is constant.
- Aggregate risk-aversion $\bar{\alpha} = 1 / \sum_{i=1}^{n} \frac{1}{\alpha_i}$; aggregate time-preference $\bar{\beta} = \sum_{i=1}^{n} \frac{\bar{\alpha}}{\alpha_i} \beta_i$.
- Exact equilibrium.

**Theorem**

Assume $\bar{\beta} + \bar{\alpha} \mu_D - \frac{\bar{\alpha}^2}{2} \sigma_D^2 > 0$. The pair $(r, P_t)$ defined by

$$
r = \bar{\beta} + \bar{\alpha} \mu_D - \frac{\bar{\alpha}^2}{2} \sigma_D^2,
$$

$$
P_t = \frac{D_t}{r} + \frac{\mu_D - \bar{\alpha} \sigma_D^2}{r^2}
$$

is a market equilibrium, and the agents’ optimal policies are

$$
\hat{\theta}_t^i = \frac{\bar{\alpha}}{\alpha_i},
$$

$$
\hat{c}_t^i = \frac{\bar{\alpha}}{\alpha_i} D_t + \frac{\bar{\beta} - \beta_i}{\alpha_i} \left( t - \frac{1}{r} \right) + r \left( x_i - \frac{\bar{\alpha}}{\alpha_i} P_0 \right).
$$
Baseline Features

• Representative agent: same prices and rates as in equilibrium with one agent with preferences \((\bar{\alpha}, \bar{\beta})\).
• Markov Equilibrium: optimal policies and price dynamics depend only on \((t, D_t)\).
• No-trade Equilibrium: number of shares constant for all agents.
• Consumption affected by relative impatience and relative initial wealth.
Second-Order Equilibrium

Theorem

Assume \( \bar{\alpha} = \bar{\beta} + \bar{\alpha}\mu_D - \frac{\bar{\alpha}^2}{2}\sigma_D^2 > 0 \). A second-order equilibrium is given by

\[
r_t = \bar{r} + \varepsilon \bar{\alpha} \sigma_z Z_t - \frac{\varepsilon^2}{2} \sum_{j=1}^{n} \frac{\bar{\alpha}}{\alpha_j} (m_{t,j}^2,)^2,
\]

\[
P_t = D_t \tilde{C}_t + (\mu_D - \bar{\alpha}\sigma_D^2) \tilde{L}_t + \varepsilon \frac{1}{\bar{r}(a + \bar{r})}\sigma_z Z_t - \varepsilon^2 \frac{2\bar{\alpha}\sigma_Z^2}{\bar{r}(a + \bar{r})(2a + \bar{r})} \left( Z_t^2 + \frac{1}{\bar{r}} \right),
\]

\[
\hat{\theta}_t = \frac{\bar{\alpha}}{\alpha_i} + \varepsilon^2 \frac{\bar{r}^2}{\sigma_D^2} \frac{p m_{t,1}^i}{\alpha_i \bar{r}},
\]

\[
\hat{c}_i = \frac{\bar{\alpha}}{\alpha_i} D_t + \frac{1}{C_0^i} \left( x_i - \frac{\bar{\alpha}}{\alpha_i} P_0 \right) + \frac{\bar{\beta} - \beta_i}{\alpha_i} \left( t - \frac{L_0^i}{C_0^i} \right)
\]

\[
- \frac{\varepsilon}{\alpha_i} \int_0^t m_{s}^i dW_s^\mu - \frac{\varepsilon^2}{\alpha_i} \left( \int_0^t l_{s}^i ds + \int_0^t m_{s}^i dW_s^\mu + \int_0^t n_{s}^i dW_s^D \right) + \frac{1}{\alpha_i} \frac{K_0^i}{C_0^i},
\]

where the quantities \( m_{t,1}^i, p m_{t}^i, C_t, L_t, \tilde{C}_t, \tilde{L}_t, l_{t}^i, m_{t}^i, n_{t}^i, K_t \) can be expressed in closed form.
Consumption Plan

• Let $C_i^t$ be the indifference prices (for agent $i$) of an annuity that pays 1 in perpetuity and $L_i^t$ that of an annuity that pays $s$ at time $s > t$.

• Consumption:

$$
\hat{c}_i^t = \frac{\bar{\alpha}}{\alpha_i} D_t + \frac{\bar{\beta} - \beta_i}{\alpha_i} \left( t - \frac{L_i^t}{C_i^t} \right) + \left( x_i - \frac{\bar{\alpha}}{\alpha_i} P_0 \right) \frac{1}{C_i^t} \\
-\varepsilon \frac{1}{\alpha_i} \int_0^t m_{s,i}^1 dW_s^\mu + \varepsilon^2 R_t,
$$

$$
\frac{m_{t,i}^1}{\alpha_i} = -\frac{\bar{\alpha}\sigma_z}{a + \bar{r}} \left( \frac{\bar{\beta} - \beta_i}{\alpha_i} \left( t + \frac{1}{a + \bar{r}} \right) + \left( x_i - \frac{\bar{\alpha}}{\alpha_i} P_0 \right) \bar{r} \right)
$$

• If annuities were tradeable, optimal consumption would be first three terms. The consumption plan in the complete market cannot be financed when the market is incomplete.
Mimicking Non-tradeable Assets

- Define $Q^i_t := \frac{\bar{\beta} - \beta^i_{\alpha}}{\alpha_i} \left( L^i_t - \frac{L^i_0}{C^i_0} C^i_t \right) + \frac{1}{C^i_0} \left( x^i_t - \frac{\bar{\alpha}}{\alpha_i} P^i_0 \right) C^i_t$. $Q^i_t$ is the value of the “desired” position in long-term bonds.

- Agent $i$ tries to replicate this position with the risk-free asset:

$$X^i_t - \hat{\theta}^i P_t = Q^i_t - \int_0^t \frac{d\langle Q^i, W^\mu \rangle_s}{ds} \, dW^\mu_s + O(\varepsilon^2).$$

- Interest-rate proceeds on excess lending are consumed:

$$\hat{c}^i_t = \hat{c}_t^{i,\text{comp}} - \bar{r} \int_0^t \frac{d\langle Q^i, W^\mu \rangle_s}{ds} \, dW^\mu_s + O(\varepsilon^2).$$

- Noisy tracking error in the replication leads to uncertainty in consumption.
Consumption: Information and Heterogeneity

- Consumption depends on the history of $W^\mu$-shocks, as tracking errors cumulate. It is not Markov in the natural state variables $(t, D_t, z_t)$.

- Agents who deviate most from the average (high $|\bar{\beta} - \beta_i|$, high $|\frac{\alpha_i}{\alpha_i} P_0 - x_i|$) have a stronger risk-sharing demand.

- These agents have a larger position in the risk-free asset, face more volatile tracking errors, and therefore higher consumption uncertainty.

- Heterogeneity matters.
Dynamic Trading

- Trading arises. $\hat{\theta}^i$ not constant.

$$
\hat{\theta}_t^i = \frac{\bar{\alpha}}{\alpha_i} + \epsilon^2 \frac{\bar{r}}{\sigma_D^2} \frac{\sigma_z}{\bar{r}(a + \bar{r})} \left( -\bar{\alpha}D_t - (\bar{\mu} - \bar{\alpha}\sigma_D^2) \left( \frac{1}{\bar{r}} + \frac{1}{a + \bar{r}} \right) \bar{\alpha} + 1 \right) \cdot \frac{m_t^{1,i}}{\alpha_i}
$$

- Why do agents trade? To hedge fundamental shocks in an incomplete market. Hedging strategies cannot account for all future contingencies and need updating.

- In the literature, agents trade because of overlapping generations, idiosyncratic labor income, noise traders. Features absent in this model.
**Reward for Consumption Risk**

The second order adjustment to consumption consists of three terms:

1) $dW_t^\mu$-term, as at first order;

2) dividend shock $dW_t^D$-term, proportional to the number of shares $\hat{\theta}_i^t$ held at time $t$,

3) each agent wants a reward for additional consumption volatility ($m_t^{i,1}$):

$$
\frac{\varepsilon^2}{2\alpha_i} \int_0^t \left( (m_s^{i,1})^2 - \sum_{j=1}^N \frac{\bar{\alpha}}{\alpha_j} (m_s^{j,1})^2 \right) ds
$$

- But not everyone can get it: sum must be zero.

- Instead, only atypical agents with *above-average* consumption volatility are rewarded.
Interest Rate - First Order

• Interest rate has expansion

\[ r_t = \bar{\beta} + \bar{\alpha}(\bar{\mu} + \varepsilon \sigma_z z_t) - \frac{\bar{\alpha}^2}{2} \sigma_D^2 - \frac{\varepsilon^2}{2} \sum_{j=1}^{n} \frac{\bar{\alpha}}{\alpha_j} (m^{1,j}_t)^2, \]

\[ m^{1,i}_t = -\frac{\bar{\alpha} \sigma_z}{a + \bar{r}} \left( (\bar{\beta} - \beta_i) \left( t + \frac{1}{a + \bar{r}} \right) + \left( x_i - \frac{\bar{\alpha}}{\alpha_i} P_0 \right) \alpha_i \bar{r} \right) \]

• Growth rate of aggregate consumption is stochastic: \( \bar{\mu} + \varepsilon \sigma_z z_t. \)

• Procyclical interest rate. At first order, follows a Vasicek model (Zapatero, Goldstein (1996))
Interest Rate - Second Order

\[- \frac{\varepsilon^2}{2} \sum_{j=1}^{n} \frac{\bar{\alpha}}{\alpha_j} (m_t^{1,j})^2, \quad m_t^{1,i} = -\frac{\bar{\alpha} \sigma_z}{a + \bar{r}} \left( (\bar{\beta} - \beta_i) \left( t + \frac{1}{a + \bar{r}} \right) + \left( x_i - \frac{\bar{\alpha}}{\alpha_i} P_0 \right) \alpha_i \bar{r} \right) \]

- Precautionary savings effect: \((m_t^{1,j})^2\) is proportional to the quadratic variation \(d\langle c_j \rangle_t\).
- Atypical agents face high consumption uncertainty. Heterogeneity reduces the interest rate.
- Autarchic endowments \((x_i = \frac{\bar{\alpha}}{\alpha_i} P_0)\) lead to a rate-decrease proportional to the cross-sectional variance of time-preference, weighted inversely to risk aversion:

\[\sum_{i=1}^{n} \frac{\bar{\alpha}}{\alpha_i} (\bar{\beta} - \beta_i)^2\]

- With homogeneous time-preference, rate-decrease proportional to

\[\sum_{i=1}^{n} \alpha_i \left( x_i - \frac{\bar{\alpha}}{\alpha_i} P_0 \right)^2\]

- The precautionary savings effect vanishes with homogeneous preferences and capital allocations.
At first order, the stock price is

\[ P_t = D_t C_i^t + (\bar{\mu} - \bar{\alpha}\sigma_D^2)(L_t^i - tC_i^t) + \varepsilon \frac{1}{\bar{r}(a + \bar{r})}\sigma_z z_t + O(\varepsilon^2). \]

Good times \((z_t \text{ high})\) have twofold effect:
1. decrease prices by discount effect;
2. Increase prices by dividend growth.

Market incompleteness does not alter prices at first order.
Discounting Effect

- Adjustment to discounting affects stock prices.

- Heterogeneity increases stock prices, because it lowers the interest rate.

- The stock price is

\[
P_t = P_t^{\text{comp}} + \varepsilon^2 \cdot \left[ D_t \left( \frac{1}{2} \sum_j \frac{\bar{\alpha}}{\alpha_j} M_t^{j, C} \right) + (\bar{\mu} - \bar{\alpha} \sigma_D^2) \left( \frac{1}{2} \sum_j \frac{\bar{\alpha}}{\alpha_j} M_t^{j, L} \right) \right],
\]

where \( M_t^{j, C} = \int_t^\infty e^{-\bar{\tau}(s-t)} \int_t^s (m_u^{j,1})^2 du ds \) and \( M_t^{j, L} = \int_t^\infty e^{-\bar{\tau}(s-t)} (s-t) \int_t^s (m_u^{j,1})^2 du ds \).
Shadow Price of Long-Term Bond

\[ C_t^i = \frac{1}{\bar{r}} - \varepsilon \frac{\bar{\alpha}}{\bar{r}(a + \bar{r})} \sigma_z z_t + \varepsilon^2 \left[ \frac{\bar{\alpha}^2 \sigma_z^2}{\bar{r}(a + \bar{r})(2a + \bar{r})} \left( z_t^2 + \frac{1}{\bar{r}} \right) + \frac{1}{2} \sum_{j=1}^{N} \frac{\bar{\alpha}}{\alpha_j} M_t^{j,C} - \bar{\alpha} \frac{\sigma_z}{a} \int_t^\infty \frac{ae^{-r(u-t)}}{r(a + r)} m_u^{j,1} du \right], \]

where \( M_t^{j,C} = \int_t^\infty e^{-\bar{r}(s-t)} \int_t^s (m_u^{j,1})^2 du ds. \)

- First three terms, convexity effect:
  price of the bond if the market were complete.

- Fourth term, effect of agents’ heterogeneity.

- Fifth term, agent-specific.
  From the individual marginal utility from the bond’s cash flow.
  Lenders value bond more.
Heuristic Derivation

- Marginal utility of $i$-th agent:
  \[ e^{-\beta_i t - \alpha_i \hat{c}_i^i} = y_i M_t^i \]

- Stochastic discount factor for $\varepsilon = 0$ is $M_t = e^{-\bar{\beta} t - \bar{\alpha} (D_t - D_0)}$.

- Set $M_t^i = e^{-\bar{\beta} t - \bar{\alpha} (D_t - D_0) + Y_t^i}$, where $\lim_{\varepsilon \downarrow 0} Y_t^i = 0$
  \[ \frac{1}{\alpha_i} \left( -\bar{\beta} t - \bar{\alpha} (D_t - D_0) + Y_t^i \right) = -\frac{\beta_i}{\alpha_i} t - \hat{c}_t^i - \frac{1}{\alpha_i} \log y_i, \]

- Summing over agents, $\sum_i \frac{1}{\alpha_i} Y_t^i = - \sum_i \frac{1}{\alpha_i} \log y_i - D_0$ and, hence,
  \[ \sum_i \frac{1}{\alpha_i} dY_t^i = 0. \]

- Assume $Y_t^i = \varepsilon Y_t^{i,1} + o(\varepsilon)$, where
  \[ dY_t^{i,1} = I_t^{i,1} dt + m_t^{i,1} dW_t^\mu + n_t^{i,1} dW_t^D. \]
Agreement on Traded Assets

- Interest rate is the negative growth rate of $M_t^i$:

$$r_t = \bar{\beta} + \bar{\alpha} (\bar{\mu} + \epsilon \sigma z_t) - \frac{1}{2} \bar{\alpha}^2 \sigma_D^2 - \frac{1}{2} \epsilon^2 (m_t^{i,1})^2 - \frac{1}{2} \epsilon^2 (n_t^{i,1})^2 + \epsilon \bar{\alpha} \sigma_D n_t^{i,1}.$$

- Ensuring that it is independent of $i$ implies $-l_t^{i,1} + \bar{\alpha} \sigma_D n_t^{i,1} = g_t$.

- Since $0 = \sum_i \frac{1}{\bar{\alpha}_i} (-l_t^{i,1} + \bar{\alpha} \sigma_D n_t^{i,1}) = \frac{g_t}{\bar{\alpha}}$,

$$-l_t^{i,1} + \bar{\alpha} \sigma_D n_t^{i,1} = 0.$$

- Similar argument on the stock price leads to $n_t^{i,1} = 0$. Hence,

$$l_t^{i,1} = n_t^{i,1} = 0.$$

- How to identify $m_t^{i,1}$?
Replicability of Consumption Stream

- Consumption stream from first-order condition:

$$\hat{c}_t^i = \frac{\bar{\alpha}}{\alpha_i} (D_t - D_0) + \frac{\bar{\beta} - \beta_i}{\alpha_i} t - \frac{1}{\alpha_i} \log y_i - \frac{1}{\alpha_i} Y_t^i$$

- Price both sides and set:

$$L_t^i = E_t \left[ \int_t^\infty s \frac{M_t^i}{M_i} ds \right],$$
$$C_t^i = E_t \left[ \int_t^\infty \frac{M_t^i}{M_i} ds \right],$$
$$K_t^i = E_t \left[ \int_t^\infty Y_t^i \frac{M_t^i}{M_i} ds \right].$$

$$X_t^i = \frac{\bar{\alpha}}{\alpha_i} P_t + \frac{\bar{\beta} - \beta_i}{\alpha_i} L_t^i - \frac{\bar{\alpha}}{\alpha_i} D_0 C_t^i - \frac{1}{\alpha_i} (\log y_i) C_t^i - \frac{1}{\alpha_i} K_t^i.$$

- Must satisfy budget equation

$$dX_t^i = \theta_t^i (dP_t + D_t dt) + r_t (X_t^i - \theta_t^i P_t) dt - c_t^i dt$$

- Matching diffusion coefficients of $X_t^i$ from both equations yields expressions for $m_t^{i,1}$ and $\theta_t^{i,1}$.
Conclusion

• Closed-form general equilibrium with unhedgeable fundamental shocks and heterogeneous preferences.

• Perturbation around complete baseline measure.

• Uncover effects proportional to unhedegable shock size and its variance.

• Preference heterogeneity affects consumption and prices, dynamic trading, consumption premium.
Thank You!
Multiple Assets

- Model extends to several stocks:

\[ dD_t = (\bar{\mu} + \varepsilon \sigma z_t) dt + \sigma_D dW^D_t, \]

\[ dz_t = -az_t dt + \rho dW^D_t + \sqrt{1 + \rho^2} d\tilde{W}_t^\mu. \]

where \( W^D \) and \( W^\mu \) are \( k \)-dimensional independent Brownian Motions, and \( \sigma z, A \in \mathbb{R}^{k \times k} \).

- Define \( S_t^i := E_t \left[ \int_t^\infty e^{-\int_s^t r_u du} D^i_s ds \right] \) and \( R_t^i := S_t^i - P_t^i \).

- Let the market beta of the \( i \)-th asset be \( \text{Beta}_i := \frac{d\langle P^i, P^m \rangle_t}{d\langle P^m \rangle_t} \), where \( P^m_t = \sum_{i=1}^n P_t^i \).
Multiple Assets

- Cross-sectional asset pricing implication: At the first order,

\[ R^i = \beta^i R^m + \bar{\alpha} \left( \frac{1^T \sigma_D \rho}{\bar{r}^2(a + \bar{r})} \right) \left( D^i_t - \beta^i(1^T D_t) \right) \left( \frac{\sigma z^i}{1^T \sigma z} - \beta^i \right) - \frac{2a + 3\bar{r}}{\bar{r}(a + \bar{r})} \bar{\alpha} (\mu^i - \beta^i(1^T \mu)) \].

- \( \beta^i \) is the market beta of the i-th asset. CAPM holds if \( \rho = 0 \). Deviations otherwise.
- Fixed value effect from relative growth \( \mu^i - \beta^i(1^T \mu) \).
- Relative dividend \( D^i_t - \beta^i(1^T D_t) \) produces size effect. No low-dimensional factor models.
First-Order Equilibrium – Definition

Definition

A first-order equilibrium is a family \( \{ r_t(\varepsilon), P_t(\varepsilon), (\hat{c}^i_t(\varepsilon))_{1 \leq i \leq n}, (\hat{\theta}^i_t(\varepsilon))_{1 \leq i \leq n} \}_{0 \leq \varepsilon \leq \bar{\varepsilon}} \) such that (henceforth, unless ambiguity arises, the dependence on \( \varepsilon \) is omitted)

1. \( P_t(\varepsilon)_{0 \leq \varepsilon \leq \bar{\varepsilon}} \) is a continuous semimartingale and \( r_t(\varepsilon)_{0 \leq \varepsilon \leq \bar{\varepsilon}} \) an adapted, integrable process;

2. \( \{ \hat{c}^i_t, \hat{\theta}^i_t \}_{0 \leq \varepsilon \leq \bar{\varepsilon}} \) is admissible for all \( 1 \leq i \leq n \);

3. For every \( \varepsilon \) the market clearing conditions hold:

\[
\sum_{i=1}^{n} \hat{\theta}^i_t = 1, \quad \sum_{i=1}^{n} (X^i_t - \hat{\theta}^i_t P_t) = 0, \quad \sum_{i=1}^{n} \hat{c}^i_t = D_t;
\]

4. For every agent \( i \), the family of strategies \( \{ \hat{c}^i_t, \hat{\theta}^i_t \}_{0 \leq \varepsilon \leq \bar{\varepsilon}} \) is first-order optimal in \( \varepsilon \), i.e.,

\[
E \left[ \int_0^\infty \frac{1}{1 - \alpha_i} e^{-\beta_i t - \alpha_i \hat{c}^i_t} dt \right] \geq \sup_{(c_t, \theta_t) \in A} E \left[ \int_0^\infty \frac{1}{1 - \alpha_i} e^{-\beta_i t - \alpha_i c_t} dt \right] + o(\varepsilon).
\]
Admissible Strategy

Definition

Let $P_t(\varepsilon)$ be a continuous semimartingale and $r_t(\varepsilon)$ an adapted, integrable process, for all $\varepsilon \in (0, \bar{\varepsilon})$. A family $\{c_t(\varepsilon), \theta_t(\varepsilon)\}_{0 \leq t < \infty}$ is an admissible strategy if

1. $(c_t(\varepsilon), \theta_t(\varepsilon))$ is a pair of adapted processes for all $\varepsilon \in (0, \bar{\varepsilon})$.
2. There exists $\delta > 0$ independent of $\varepsilon$ such that for any $\varepsilon \in (0, \bar{\varepsilon})$ and any $T$ the process $\theta_t(\varepsilon)$ is $(2 + \delta)$-integrable on $[0, T] \times \Omega$.
3. The process $c_t(\varepsilon)$ is bounded from below uniformly in $\varepsilon$, i.e., for some polynomial functions $q_1, q_2$ independent of $\varepsilon$, $c_t(\varepsilon) \geq q_1(t, D_t) + \int_0^t q_2(s, D_s)dW_s^D$.
4. For every $\varepsilon \in (0, \bar{\varepsilon})$ the budget equation

$$dX_t(\varepsilon) = \theta_t(\varepsilon)(dP(\varepsilon) + D_t dt) + r_t(\varepsilon)(X_t(\varepsilon) - \theta_t(\varepsilon)P_t(\varepsilon))dt - c_t(\varepsilon)dt,$$ $X_0(\varepsilon) = x_i$,

has a strong solution $X_t(\varepsilon)$ that is bounded from below uniformly in $\varepsilon$, i.e., for some polynomials $p_1, p_2, p_3$ independent of $\varepsilon$, $X_t(\varepsilon) \geq p_1(t, z_t, D_t) + \int_0^t p_2(s, z_s, D_s)dW_s^D + \int_0^t p_3(s, z_s, D_s)dW_s^\mu$, and is $(2 + \delta)$-integrable on $[0, T] \times \Omega$ for some $\delta > 0$ independent of $\varepsilon$ and any $T$. 