Optimal Dynamic Asset Allocation with Transaction Costs: The Role of Hedging Demands

Mehmet Sağlam

Carl H. Lindner College of Business
University of Cincinnati
email: mehmet.saglam@uc.edu

Joint with Pierre Collin-Dufresne (EPFL) and Kent Daniel (Columbia)

Princeton University, SCFE 2023
Literature on Frictionless Portfolio Choice

- Markowitz’s (1952) one-period mean-variance portfolio choice is still widely popular among practitioners.
- Merton (1969, 1971) and Cox-Huang (1989) introduce hedging demand:
  → In dynamic setting, it may be optimal to deviate from instantaneous mean-variance portfolio to hedge against future changes of mean-variance frontier.
- Academic work on return predictability emphasizes empirical relevance of hedging demand:
  → Hundreds of anomalies have been discovered in the last three decades.
  → Institutions trade with large orders. Realistic asset allocation needs to account for price impact (slippage) costs.
A Concrete Example

- As an example consider optimal combination of three signals for large cross-section of individual stocks
  - Short term reversal (REV: half-life of 5 days)
  - Momentum (MOM: half-life 150 days)
  - Value (VAL: long-term reversal with a half-life of 700 days)

- Each stock will have a specific exposure to REV, MOM, and VAL. These factors will decay at different rates and are clearly not independent.

- How do we operationalize the optimal trade-off when there are transaction costs?
  - If trade more often, expect to capture more alpha, but pay more transaction costs.
  - If trade less often, may not benefit from fast signals.
    → Should we trade more/less aggressively when signals decay faster/slower?
  - Trading more frequently reduces tracking error, but increases t-costs.
No analytical solution in the prior literature

• Studies combining predictability in returns and t-costs typically limited to one or two risky assets and use numerical solutions [Balduzzi and Lynch, 1999; Lynch and Tan, 2011; Longstaff, 2001]
The Linear Quadratic Framework

- In the **Linear-Quadratic (LQ)** framework, explicit closed-form solution can be obtained for optimal portfolio choice with many stocks, many predictors, and quadratic t-costs (i.e., linear price impact) [Garleanu and Pedersen, 2013 (~600 citations); Collin-Dufresne, Daniel, Saglam, 2020]

- The **LQ** framework relies on ad-hoc conditional mean-variance (CMV) objective function. The solution is the classical Markowitz portfolio in the absence of t-costs.

- This paper:
  - Standard preferences which nest (micro-found) this LQ-objective function.
  - Explicit portfolio choice solution for general (non-myopic) objective function with quadratic transaction costs
  - How important are **hedging demands** for portfolio choice with transaction costs?
Classical Markowitz portfolio

- One-period problem with quadratic risk penalty:

\[
\max_{n_t} \left\{ n_t^\top \mu x_t - \frac{\gamma}{2} n_t^\top \Sigma n_t \right\}
\]

- \( n_t \) is number of shares to hold
- \( \mu \) is \( N \times K \) factor loading matrix
- \( x_t \) is \( K \times 1 \) vector of predictors
- \( \Sigma \) is the covariance matrix of dollar returns

- The optimal portfolio is the conditional mean-variance efficient (CMVE) Markowitz portfolio:

\[
n_t^* = CMVE_t = (\gamma \Sigma)^{-1} \mu x_t
\]

- When would you deviate from this portfolio?
  - Transaction costs
  - Non-myopic long-term investor
Introduce Quadratic Trading Costs

• A one period problem with quadratic transaction costs:

\[
\max_{n_t} \left\{ n_t^\top \mu x_t - \frac{h}{2} (n_t - n_{t-1})^\top \Lambda (n_t - n_{t-1}) - \frac{\gamma}{2} n_t^\top \Sigma n_t \right\}
\]

- Quadratic TC

• \( h \) is transaction cost multiplier
• \( \Lambda \) is price impact matrix.

• If \( \Lambda = \lambda \Sigma \), the optimal position is

\[
n_t^* = (1 - \tau) n_{t-1} + \tau (\gamma \Sigma)^{-1} \mu x_t
\]

⇒ Account for trading costs by partially trading towards Markowitz portfolio with trading speed:

\[
\tau = \frac{1}{1 + h\lambda / \gamma}
\]

• Practitioners typically optimize over the trading speed parameter (\( h \)) to maximize backtest performance.
Linear-Quadratic Dynamic Model

- $N$-vector of stock price $S_t$ and $K$-vector of predictor $x_t$ have dynamics:

$$
\begin{align*}
\quad \quad dS_t &= \mu x_t dt + \sigma_s dZ^s_t \\
\quad \quad dx_t &= -\kappa x_t dt + \sigma_x dZ^x_t + \sigma_{xs} dZ^s_t
\end{align*}
$$

- $\Sigma = \sigma_s \sigma_s^\top$ is the covariance matrix of dollar returns
- $\kappa$ captures mean-reversion parameters
- Return shocks can be correlated with predictor shocks: $\Sigma_{sx} = \sigma_s \sigma_{xs}^\top$

**Example:** In one-dimensional setting, let $x_t$ be given by the dividend yield, i.e., $x_t = \frac{D}{S_t}$. Then,

- $\Sigma_{sx} < 0$ as positive shocks to dividend yield increase expected returns but are contemporaneously negatively correlated with returns.

- A bad news ($dS < 0$) is not very bad, as it increases expected returns
Trading Costs & CMV Objective Function

- **Wealth dynamics with position vector** $n_t$:

  $$dW_t = n_t^T dS_t - \frac{1}{2} \theta_t^T \Lambda \theta_t dt$$

  $$= n_t^T (\mu x_t) dt + n_t^T \sigma_s dZ_s(t) - \frac{1}{2} \theta_t^T \Lambda \theta_t dt$$

  $$dn_t = \theta_t dt$$

- **CMV-based objective function**:

  $$\max_{\theta} \mathbb{E} \left[ \int_0^T \left\{ dW_t - \frac{\gamma}{2} dW_t^2 \right\} \right]$$

  $$= \max_{\theta} \mathbb{E} \left[ \int_0^T \left\{ n_t^T (\mu x_t) - \frac{1}{2} \theta_t^T \Lambda \theta_t - \frac{\gamma}{2} n_t^T \Sigma n_t \right\} dt \right]$$


CMV: Optimal Trading

- The aim portfolio can be written as

$$\text{aim}(x, t) = (\gamma \Sigma)^{-1} \int_t^T \omega_{t,u} \mu_S(t, u) du$$

- Can be interpreted as a Markowitz portfolio where we replace the expected return vector by a trading-speed weighted average of future expected returns:

$$\mu_S(t, u) = \frac{1}{dt} E_t [dS_u] = \mu e^{-\int_t^u \kappa ds} x_t$$

$$\omega_{t,u} = (\int_t^T e^{-\int_t^z \tau_s^\top ds} dz)^{-1} e^{-\int_t^u \tau_s^\top ds}$$

- Trade at a constant rate towards an aim portfolio where trading speed only depends on $\gamma \Lambda^{-1} \Sigma$. 
Further Properties of CMV Trading

- Aim portfolio equals the Markowitz (CMVE) portfolio if there are no transaction costs: $\Lambda = 0$.

- Aim portfolio equals the Markowitz (CMVE) portfolio if signals are infinitely long-lived: $\kappa = 0$.

- Trading is independent of the signal-specific risk ($\sigma_x$).

- Trading is independent of the correlation between returns and signals ($\sigma_{xs}$).

- Signals can be fully deterministic!
CARA Preferences: No Transaction Costs

• Investor maximizes CARA expected utility of terminal wealth when $\Lambda = 0$ (no t-costs):

\[
\max_{n_t} E_t \left[ -e^{-\gamma W_T} \right]
\]  

(3)

• The optimal position is given by (Merton 1971):

\[
n_t = (\gamma \Sigma)^{-1} (\mu x_t) - \Sigma^{-1} \Sigma_{sx} (d_t x_t)
\]  

(4)

Markowitz

Hedging Demand

where $\Sigma = \sigma_s \sigma_s^\top$ and $\Sigma_{sx} = \sigma_s \sigma_{xs}^\top$.

$\Rightarrow$ If zero correlation between returns and expected returns (i.e., $\Sigma_{sx} = 0$) it is optimal to hold the CMVE Markowitz portfolio.

• Example: $\Sigma_{sx} < 0$ for $x_t = D/S_t$ (dividend yield) leading to larger target portfolio.
The general objective function

- Consider solution \((H_t, \sigma_{H,s}, \sigma_{H,x})\) to the recursive equation:

\[
H_t = E_t[W_T - \int_t^T \mu_H(H_z, \sigma_{H,s}, \sigma_{H,x})dz]
\]  \hspace{1cm} (5)

- \(H_t\) can be recast as certainty equivalent of source-dependent stochastic differential utility agent with CARA coefficient \(\gamma (\gamma_x)\) towards \(Z^s (Z^x)\) shocks.
  - When \(\gamma_x = \gamma\) it nests CARA expected utility:
    \[
    H_t = -\frac{1}{\gamma} \log(E_t[e^{-\gamma W_T}]).
    \] \hspace{1cm} (6)
  - When \(\gamma_x \sigma_x \to 0\) and \(\sigma_{xs} = 0\) it nests CMV:
    \[
    H_t = W_t + E_t \left[ \int_t^T \left\{ n_u^\top (\mu_{x_u}) - \frac{1}{2} \theta_u^\top \Lambda \theta_u - \frac{\gamma}{2} n_u^\top \Sigma n_u \right\} du \right].
    \] \hspace{1cm} (7)

\(\rightarrow\) The CMV investor is risk-neutral with respect to changes in the investment opportunity set. If \(\sigma_x = \sigma_{xs} = 0\), then CARA and CMV are identical.
The Optimal Trading Policy

• In the presence of transaction costs ($\Lambda > 0$), then the optimal trading rule is given by

$$dn_t = \tau_t (aim(x_t, t) - n_t) \, dt$$  \hspace{1cm} (8)

$$\tau_t = \Lambda^{-1} Q(t)$$  \hspace{1cm} (9)

$$aim(x, t) = Q(t)^{-1}(q_0(t) + q(t)^{\top} x)$$  \hspace{1cm} (10)

where $Q, q, q_0$ solve system of Riccatti-style ODEs.

• Why is this important? Comparing solution to CARA with solution to CMV, we can quantify the impact of hedging demand on portfolio choice with transaction costs.
Results extend to Random Horizon

- Consider solution \( (H_t, \sigma_{H,s}, \sigma_{H,x}) \) to the recursive equation with random horizon \( T \) (Poisson with intensity \( \rho \)) where \( H_t \) equals:

\[
E_t \left[ W_T - \int_t^T \left\{ \frac{1}{2} \gamma ||\sigma_{H,s}||^2 + \frac{1}{2} \gamma_x ||\sigma_{H,x}||^2 + \rho \left( W_s - H_s - \frac{1 - e^{-\gamma T (W_s - H_s -)}}{\gamma_T} \right) \right\} ds \right]
\]

- Then \( H_t \) can be recast as the certainty equivalent of source dependent utility agent with CARA coefficient \( \gamma \) toward \( Z^s \), \( \gamma_x \) towards \( Z^x \), and \( \gamma_T \) towards horizon arrival \( 1_{\{T \leq t\}} \).
  - When \( \gamma_T = \gamma_x = \gamma \), it nests CARA expected utility:
    \[
    H_t = -\frac{1}{\gamma} \log(E_t[e^{-\gamma W_T}]).
    \] (11)
  - When \( \gamma_T = \gamma_x \sigma_x \to 0 \) and \( \sigma_{xs} = 0 \), it nests the discounted CMV objective function:
    \[
    H_t = W_t + E_t \left[ \int_t^\infty e^{-\rho(u-t)} \left\{ n_u^\top (\mu x_u) - \frac{1}{2} \theta_u^\top \Lambda \theta_u - \frac{\gamma}{2} n_u^\top \Sigma n_u \right\} du \right].
    \] (12)
Illustration with One Asset and One Signal

Figure: Aim Portfolios and Trading Speed (Persistent Signal)
Illustration with One Asset and One Signal

Figure: Aim Portfolios and Trading Speed (Fast-decaying Signal)
Real World Application with a Trading Signal

- From publicly available trade data set (TAQ), it is possible to identify retail trades.

- Retail orders are routed to high-frequency market makers (e.g., Citadel, Virtu, Two Sigma).

- These market makers provide a trivial price improvement: If the best price to buy (ask) a stock is $100, the retail trader gets a price of $99.9999.

- These trades are reported to TAQ with a certain venue identifier, ‘D’.

- Simple algorithm: Take all ‘D’ transactions. If
  - $100 \times \text{mod}(\text{Price}, 0.01) \in [0.6, 1)$, it is buy retail order.
  - $100 \times \text{mod}(\text{Price}, 0.01) \in (0, 0.4)$, it is sell retail order.
Wisdom of Crowds

- Let $N_{i,t}^b$ ($N_{i,t}^s$) be the number of retail buy (sell) orders on stock $i$ and day $t$, using the retail trade classification. Our return predictor at the stock level is then given by

$$x_{i,t} = \frac{N_{i,t}^b - N_{i,t}^s}{N_{i,t}^b + N_{i,t}^s}.$$

- We take two stocks, JNJ and XOM, and estimate the parameters of the model using data between 2014–2016.

- We run the following regressions to test the empirical relation between net retail order flow and subsequent daily returns:

$$r_{1,t+1} = \beta_1 x_{1,t} + \epsilon_{1,t+1}$$

$$r_{2,t+1} = \beta_2 x_{2,t} + \epsilon_{2,t+1}$$
### Regression Results

<table>
<thead>
<tr>
<th>Dependent variable:</th>
<th>$r_{1,t+1}$</th>
<th>$r_{2,t+1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_{1,t}$ (JNJ)</td>
<td>0.0058*</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.0032)</td>
<td></td>
</tr>
<tr>
<td>$x_{2,t}$ (XOM)</td>
<td></td>
<td>0.0057*</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.0034)</td>
</tr>
<tr>
<td>Constant</td>
<td>0.0002</td>
<td>0.00005</td>
</tr>
<tr>
<td></td>
<td>(0.0004)</td>
<td>(0.0004)</td>
</tr>
<tr>
<td>Observations</td>
<td>755</td>
<td>755</td>
</tr>
<tr>
<td>Adjusted $R^2$</td>
<td>0.01</td>
<td>0.01</td>
</tr>
</tbody>
</table>

Table: Net retail order flow imbalance and subsequent returns
Mean-Reversion in Predictors

<table>
<thead>
<tr>
<th></th>
<th>$\Delta x_{1,t+1}$</th>
<th>$\Delta x_{2,t+1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_{1,t}$ (JNJ)</td>
<td>$-0.4862^{***}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.031)</td>
<td></td>
</tr>
<tr>
<td>$x_{2,t}$ (XOM)</td>
<td></td>
<td>$-0.3469^{***}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.028)</td>
</tr>
<tr>
<td>Constant</td>
<td>0.020$^{***}$</td>
<td>0.002</td>
</tr>
<tr>
<td></td>
<td>(0.004)</td>
<td>(0.004)</td>
</tr>
<tr>
<td>Observations</td>
<td>755</td>
<td>755</td>
</tr>
<tr>
<td>Adjusted $R^2$</td>
<td>0.243</td>
<td>0.173</td>
</tr>
</tbody>
</table>

Table: AR(1) regressions of net retail order flow imbalance
Estimation Findings

- For JNJ stock, the half-life of the net retail order imbalance signal is approximately one-day while for XOM stock the half-life of the signal is 1.6 days.

- We can calibrate $\Sigma$, $\Sigma_x$ and $\Sigma_{sx}$ from this in-sample data.

- Diagonal entries in $\Sigma_{sx}$ are negative implying that when there is a positive shock to the price of either JNJ or XOM, there will be less retail buying activity on these stocks contemporaneously.

- This is consistent with a contrarian trading strategy at the aggregate retail level, i.e., retail traders tend to sell (buy) a stock with positive (negative) daily returns.
Estimation of the transaction cost parameters

- Use institutional large order data statistics from the execution desk of a large investment bank
  - Execution data covers S&P 500 stocks between January 2011 and December 2012.
  - Total number of orders is 81,744 with an average size of approximately $1 million.
  - The average participation rate of the order, the ratio of the order size to the total volume realized in the market, is approximately 6%.
- The standard measure of institutional trading cost is implementation shortfall:

\[
IS_i = D_i \frac{P_{avg}^i - P_{i,0}^i}{P_{i,0}^i},
\]  

(13)

- \( D_i \) is the direction of the trade.
- \( P_{i,0}^i \) is the initial mid-quote price of the security
- \( P_{avg}^i \) is the average execution price of the parent-order
Price Impact Estimation (Top 50 stocks)

\[ IS_i = \theta \frac{Q_i}{DayVlm_i} + \epsilon_i \]

| **Dependent variable:** IS (bps) |
|-----------------|-----------------|
| \( \frac{Q}{DayVlm} \)    | 363.20***       |
| Constant         | 0.52            |

(91.66) (1.12)

Observations 16,532
Adjusted \( R^2 \) 0.001

Table: Estimation of the price impact from institutional order data set
Out-of-sample Performance

- We test the trading performance of CARA and CMV policies in the next three years, 2017-2019.
- Trading horizon is 20 days. Corresponds to 38 non-overlapping samples.
- At the beginning of each interval we start with $0 and trade according to a trading rule (CARA vs CMV).
- Record the stock position and the total accumulated wealth net of t-costs every day.
- Compute the certainty equivalent, and average utility across all samples.
- CARA statistically dominates CMV in terms of mean utility.
- Results:

<table>
<thead>
<tr>
<th>Statistic</th>
<th>CARA</th>
<th>CMV</th>
<th>Diff</th>
</tr>
</thead>
<tbody>
<tr>
<td>CE</td>
<td>130.49</td>
<td>99.94</td>
<td></td>
</tr>
<tr>
<td>Avg Util</td>
<td>-0.9974</td>
<td>-0.9980</td>
<td>$6.1 \times 10^{-4}$</td>
</tr>
<tr>
<td>S.E</td>
<td>0.0049</td>
<td>0.0051</td>
<td>$2.6 \times 10^{-4}$</td>
</tr>
</tbody>
</table>
Managing transaction costs is key driver
Conclusion

- We propose a set of preferences based on stochastic differential utility with source-dependent risk-aversion, which nest the widely used conditional mean-variance and CARA expected utility.
- We derive an explicit solution for the portfolio choice problem in the presence of quadratic t-costs with arbitrary number of stocks and predictability in returns in terms of an optimal aim portfolio and trading speed.
- We show that, for a CARA investor, the hedging demand has large effect on optimal aim portfolio and trading speed, especially when the correlation between stock return and predictor is negative.
- In a realistic calibration where we use aggregate retail trading to predict expected returns, we show that hedging demands significantly affect strategy performance.